ON NOETHER SYMMETRIES AND CONSERVATION LAWS

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This thesis, written by **AHMAD MAHADI MUGBIL AHMAD** under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS.**

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عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

To my parents

wife

children

sister and brothers

who waited patiently for me to come out of my study

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Thesis Abstract

Name: Ahmad Mahadi Mugbil Ahmad

Title: On Noether Symmetries and Conservation Laws

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Noether symmetries provide conservation laws that are admitted by Lagrangians representing physical systems. For differential equations possessing Lagrangians these symmetries are obtained by the invariance of the corresponding action integral. In this thesis we find Noether symmetries and conserved vectors for a Lagrangian constructed from a metric that represents Milne model and is of interest in general relativity. For completeness, we construct wave equation on this metric, obtain its Lie symmetries and compare them with spacetime isometries.

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ملخص الرسالة

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توفر تناظرات نويثرقوانيين الحفظ التي تقبلها اللاجرانجيات الممثلة للأ نظمة الفزيائية. بالنسبة للمعادلات التفاضلية التي تمتلك لاجرانجيات يتم الحصول على هذه التناظرات من عدم تغير التكامل التأثيري المقابل.

في هذة الرسالة سنقوم بإيجاد تناظرات نيوثر والكميات المحفوظة للاجرانجي ناشئ من مقاسة تمثل نموذج ميلن وتحظى باهتمام في النسبية العامة.واستكمالاً لذلك ننشئ معادلة الموجة على هذة المقاسة ونسعى للحصول على تناظرات لي الخاصة بها ثم نقارنها مع تقايسات الزمكان.

ماجستير في العلوم

جامعـــــــة الملك فهـــــد للبترول والمعـــــادن

الظهران - المملكة العربية السعودية

Chapter 1

Introduction

It is a well-known fact that solving equations has played a pivotal role in the history of mathematics. For an example solving algebraic equations of order higher than four by Galois, 1831, generated a new field in mathematics known as theory of groups. In the second half of the nineteenth century, the theory of differential equations took a similar direction. At that time, solving differential equations had become one of the important problems in applied mathematics. This was due to the increasing need of determining the functional dependencies between the variables involved in the formulas that describe several phenomena in physics and other sciences by derivatives and integrals of functions, about 200 years after Newton and Leibniz introduced these concepts.

Therefore several solution techniques had been developed for certain classes of differential equations. It was quickly realized that a common feature of these techniques consist in introducing new variables such that the given differential equation is transformed into a class of known type or it reduces the order, in case of an ordinary differential equation (ODE), or reduces the number of independent variables, in case of a partial differential equation (PDE). For nonlinear differential equations (DEs), the attempts to find a solution by these techniques often terminate without any evidence whether a solution in closed form does exist or not.

In 1773, Laplace discovered his two differential invariants for the integration of scalar (1+1) linear second-order hyperbolic PDEs. Based on a preceding work by Laguerre, Malet, Halphe and others, Forsyth calculated invariants of linear differential equations under predefined classes of transformations. Later, the Norwegian mathematician Sophus Lie (1842-1899) became interested to construct a theory similar to that of Galois for differential equations using continuous groups. The resulting work later emerged as an independent theory that is known as *theory of Lie groups* and is best explained in a letter Lie wrote to A Mayer in 1874: *"In the theory of algebraic equations before Galois only these questions were proposed: Is an equation solvable by radicals, and how is it to be solved? Since Galois, among other questions proposed is this: How is an equation to be solved by radicals in the simplest way possible? I believe the time is come to make a similar progress in differential equations"* [12].

Lie discovered that the available methods of solving certain types of ODEs, such as separable, linear, homogeneous, Euler, reduced order, or exact equations, were , in fact, special cases of a general integration process based on the invariance of the ODE under *a continuous group of transformations (Lie symmetries)*. In Lie's words, he wrote in the first chapter of his book on differential equations that: *"Previous investigations on ordinary differential equations as they may be found in the custom-*

ary textbooks do not form a systematic entirety. Special integration theories have been developed, e.g., for homogeneous differential equations, for linear differential equations and other special forms of integrable differential equations. The mathematicians failed to observe, however, that the special theories may be subordinated to a general method. The foundation of this method is the concept of an infinitesimal transformation and closely related to it the concept of a one-parameter group." [27]

In Lie's work such transformations are groups that depend on continuous parameters and consist of point transformations (point symmetries), acting on the space of independent and dependent variables. For example, an autonomous system of first order ODEs defines a one-parameter Lie group of point transformations. Lie showed that the order of an ODE could be reduced by one, if it is invariant under a *one-parameter Lie group of point transformations*. He also showed that for a given linear (or nonlinear) differential equation, the admitted group of point transformations, can be determined by an explicit algorithmic scheme (*Lie's algorithm*). He also indicated that the problem of finding Lie point symmetries of a DE (ordinary or partial) reduced to solving related linear systems of *determining equations* .

Lie also showed that a point symmetry of a DE generates a one-parameter family of solutions from any known solution of a DE that is not an invariant solution arising from the symmetry. While Lie algebras (of infinitesimals) were first invented by Lie in his new approach to DEs, Killing (1847-1923) about ten years later independently introduced Lie algebras in his study of non-Euclidean geometry to classify spaces. Later Killing and Cartan provided an explicit and complete classification of Lie algebras that was considered as one of the major achievements in mathematical research at that time [24].

It is not easy to overestimate the significance of Lie's contribution to modern mathematics and science but one can say that today, Lie symmetries analysis is the most important applicable method for finding analytical solutions of nonlinear problems. Lie groups have extensive applications in variety of fields such as Mathematics, Physic, Differential geometry, and Relativity etc. In the recent few decades it has there is a renewed interest in application of Lie's theory to generate solutions of differential equations. His method leads are used to finding solutions via Lie point symmetries admitted by the DEs via both reducing the order (for ODEs) and the number of the independent variables (for PDEs).

In analyzing systems of DEs, the concept of *conservation law*, plays an important role in the study of essential properties of their solutions. Physically, these conservation laws can be identified as conservation of energy and momentum etc and have important relationship with Lie symmetries.

In 1918, Emmy Noether (1882-1935) proved in her famous paper [13], two remarkable theorems on the connection between conservation laws and symmetries for variational problems. She showed in her work that symmetries of an *action integral* lead to conservation laws for the associated geodesic equations (geodesic equations are the equations of motion and arise using a variational principal). For systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property. For example, invariance of a variational principle under time translations gives rise to conservation of energy for the solutions of the associated Euler-Lagrange equations and invariance under spatial translations implies conservation of linear momentum. While first Noether's theorem formulated a correspondence between invariance and conservation properties, the second theorem dealt with the invariance of a variational problem under the action of a group involving arbitrary functions, a case that is fundamental in general relativity and gauge theory [23].

Historically, the applications of symmetry analysis to DEs founded by Lie and Noether fell into obscurity and remained so until soon after World War II, when Birkhoff (1911-1996) brought attention to the unused applications of Lie groups to the DEs of fluid mechanics [8]. In the Soviet Union, Ovsiannikov and his students developed a systematic scheme of successfully applying Lie's algorithm to a wide range of physically important problems [25].

Today, *Lie and Noether symmetries* of DEs, is the main topic of several outstanding textbooks, including Olver [25], Bluman and Kumei [6], Bluman and Anco [5], Cantwell [8], Stephani [26], Ibragimov [17], Hydon [9], and most recently Fritz Schwarz [27], Bluman, Cheviakov, and Anco [7]. The valuable collection of results on the topic are also contained in the CRC series edited by Ibragimov [18, 19, 20].

Lie and Noether's algorithms can be applied to practically any system of ODEs and PDEs and, indeed, several symbolic manipulation computer programs have been developed for use in such studies [28], [30], [31], [32].

There are several connections between symmetries and conservation laws discussed by Bluman [11]. Situations in which DEs do not possess the usual Lagrangians, methods are developed to find relationship between symmetries and conservation laws in [10].

In order to find conservation laws for geometries of interest in Relativity, Bokhari *et al* [3, 4] investigated Noether symmetries for certain Lagrangian constructed from geometries whose metrics are of signature 2 (also called Lorentzian geometries). The connection between isometries and symmetries of geodesic equations has been discussed in many papers, more recent is the work of Feroze *et al* [2] in which this connection is discussed for maximally symmetric spaces. Extending the idea of Bokhari *et al* [3, 4], Feroze [1] showed that there exist new conserved quantities for spaces of different curvatures.

The objectives of this thesis is as follows:

Firstly, we revisit the work of Feroze. Secondly we investigate Noether symmetries of the Euler Lagrange equations of a particular metric, known as Milne model and represents an empty universe of interest in general relativity [14]. We then extend our investigations by finding and discussing the effect of the Milne metric when wave propagates through the Milne geometry.

The plan of the rest of this thesis is as follows:

Chapter 2 provides the fundamental notions, definitions, theorems and examples from the theory of Lie point symmetries. Chapter 3 is about the basic concepts, definitions and theorems required to find Noether symmetries and the corresponding conservation laws. A detailed step by step procedure to find Noether symmetries is given via an example.

In Chapter 4, we find Noether symmetries and conserved quantities for a Lagrangian corresponding to Milne metric and compare them with the isometries of this metric. This chapter also presents the wave equation on Milne metric and its Lie point symmetries with brief comparison with spacetime isometries and Noether symmetries. Finally a discussion of the results and some recommendations are given in Chapter 5.

Chapter 2

Lie Point Symmetries

In this chapter, we present the mathematical techniques that are used throughout this thesis based on references [5, 6, 8, 12, 17]. We introduce the basic theory of Lie groups, Lie algebra and Lie point symmetry of DEs. A special emphasis is put upon the infinitesimal form of Lie groups of symmetry transformations and their prolongations as a main tool for the calculation of symmetries in applications.

2.1 One-parameter Lie Groups

In differential equations, one usually uses the technique of changing variables, which in fact is a transformation of the independent or dependent variable or both. For example changing variables

$$\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y)$$

provides a transformation that maps points (x,y) into points $(\tilde{x},\tilde{y})$ and is called a point transformation. In what follows, we mean by a transformation any point transformation that depends on one or more parameters ε,

$$\tilde{x} = \tilde{x}(x,y;\varepsilon), \quad \tilde{y} = \tilde{y}(x,y;\varepsilon)$$

2.1.1 Transformations Group

Definition 1 (One-parameter transformations group)

Let $\mathbf{x} = (x^1, x^2, ..., x^n)$ *belong to the set* $\mathscr{D} \subset \mathbb{R}^n$*. For each* $\mathbf{x}$ *and the real parameter* ε *in a set* $\mathscr{S} \subset \mathbb{R}$*, define the set of one-to-one transformations*T_ε*, onto* $\mathscr{D}$

$$T_\varepsilon : \{\tilde{\mathbf{x}} = F(\mathbf{x};\varepsilon)\}. \tag{2.1}$$

The transformation T_ε *is said to be a one-parameter group of transformations with respect to the binary operation of composition* ψ *if the following conditions are satisfied:*

(i) $(\mathscr{S}, \psi)$ *is a group, namely satisfying the following axioms:*

(1) $\psi(\varepsilon_1, \varepsilon_2) \in \mathscr{S}$ *for any* ε_1 *and* $\varepsilon_2 \in \mathscr{S}$*. (closure)*

(2) $\psi(\varepsilon_1, \psi(\varepsilon_2, \varepsilon_3)) = \psi(\psi(\varepsilon_1, \varepsilon_2), \varepsilon_3)$ *for any* ε_1*,* ε_2*,* $\varepsilon_3 \in \mathscr{S}$*. (associativity)*

(3) For any ε *in* $\mathscr{S}$, $\exists$ *a unique identity element* ε_0 *in* $\mathscr{S}$ *such that*

$\psi(\varepsilon, \varepsilon_0) = \psi(\varepsilon_0, \varepsilon) = \varepsilon.$ *(identity element)*

(4) For any ε *in* $\mathscr{S}$, $\exists$ *a unique inverse element* $\varepsilon^{-1} \in \mathscr{S}$ *such that*

$\psi(\varepsilon, \varepsilon^{-1}) = \psi(\varepsilon^{-1}, \varepsilon) = \varepsilon_0.$ *(inverse element)*

(ii) For the identity element ε_0 *of the group* $(\mathscr{S}, \psi)$, $\mathbf{x}$ *is mapped to itself:*

$$T_{\varepsilon_0} : \{\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon_0) = \mathbf{x}\}$$

(iii) For any two members $T_{\varepsilon_1}, T_{\varepsilon_2}$ *of the group;*

$$T_{\varepsilon_1} : \{\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon_1)\} \ , \ T_{\varepsilon_2} : \{\tilde{\tilde{\mathbf{x}}} = F(\tilde{\mathbf{x}}; \varepsilon_2)\},$$

the result of composing them is a transformation that is a member of the group

$$T_{\varepsilon_3} : \{\tilde{\tilde{\mathbf{x}}} = F(F(\mathbf{x}; \varepsilon_1); \varepsilon_2) = F(\mathbf{x}; \varepsilon_3)\},$$

where $\varepsilon_3 = \psi(\varepsilon_1, \varepsilon_2)$.

Remark: The transformations (2.1) are the only point transformations dealt with in this thesis. Other, more general transformations (e.g., contact transformations) have been studied by Lie and others but their consideration is beyond the scope of this work. Some references are given in the bibliography [5, 6, 17].

Example 1 (Translations Group) Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and consider translations in the plane defined as

$$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) = (x + a\varepsilon, y + b\varepsilon),\ \varepsilon \in \mathbb{R},\ \ a, b \in \mathbb{R}$$

Fixed constants a, b are not both equal to zero. Repeating the transformation,

$$\tilde{\tilde{\mathbf{x}}} = (\tilde{\tilde{x}}, \tilde{\tilde{y}}) = (\tilde{x} + a\delta, \tilde{y} + b\delta) = (x + a(\varepsilon + \delta), y + b(\varepsilon + \delta)),\ \delta \in \mathbb{R}$$

leads to a transformation that is similar to the original one with the new parameter related to the first two by a composition $\psi(\varepsilon, \delta) = \varepsilon + \delta$. These transformations form a one-parameter group of transformations called one-parameter group of point translations.

Example 2 Consider the transformation

$$\begin{aligned} \tilde{x} &= -x + \alpha, \\ \tilde{y} &= y \end{aligned}$$

where $\alpha \in \mathbb{R}, (x, y) \in \mathbb{R}^2$. It can be easily checked that it is not invertible under composition

$$\begin{aligned} \tilde{\tilde{x}} &= -\tilde{x} + \beta = x - \alpha + \beta, \\ \tilde{\tilde{y}} &= -\tilde{y} = y. \end{aligned}$$

Moreover, it has no identity element, i.e., no value of α such that $\tilde{x} = x,\ \tilde{y} = y$, so it is not a transformation group.

2.1.2 Lie Transformations Group

Definition 2 (One-parameter Lie transformations group)

We say that the one-parameter transformations group

$$\tilde{\mathbf{x}} = F(\mathbf{x};\varepsilon) \tag{2.2}$$

considered in Definition 1 is a one-parameter Lie transformations group if

1. *The parameter ε is continuous, i.e. the set $\mathscr{S}$ is an interval in $\mathbb{R}$.*
2. *The function F is infinitely differentiable with respect to the real variables $\mathbf{x}$ in $\mathscr{D}$ and an analytic function of the real continuous parameter ε in $\mathscr{S}$*
3. *The function $\psi(\varepsilon_1,\varepsilon_2)$ defining the law of composition of T_ε is an analytic function of ε_1 and ε_2 in $\mathscr{S}$.*

Example 3 The group of translations in Example 1

$$\tilde{x} = x + a\varepsilon,$$
$$\tilde{y} = y + b\varepsilon,$$

where a,b are not both zeros, is a one-parameter Lie transformations group, because varying the transformation parameter ε yields the ability of moving continuously and invertibly to any point (x,y) in the plane and the function $F = (x+a\varepsilon, y+b\varepsilon)$ is infinitely differentiable with respect to $\mathbf{x} = (x,y)$. Also it is an analytic function of the parameter ε in $\mathbb{R}$ and so is $\psi(\varepsilon,\delta) = \varepsilon+\delta$.

Example 4 (Rotations Group) The group of rotations in the *x*-*y* plane

$$\tilde{x} = x\cos\varepsilon - y\sin\varepsilon,$$
$$\tilde{y} = x\sin\varepsilon + y\cos\varepsilon$$

forms a one-parameter Lie group of rotations. Clearly,

$$\tilde{\tilde{x}} = \tilde{x}\cos\delta - \tilde{y}\sin\delta = x\cos(\varepsilon+\delta) - y\sin(\varepsilon+\delta),$$
$$\tilde{\tilde{y}} = \tilde{x}\sin\delta + \tilde{y}\cos\delta = x\sin(\varepsilon+\delta) + y\cos(\varepsilon+\delta).$$

Therefore the composition law is $\psi(\varepsilon,\delta) = \varepsilon+\delta$, and $\varepsilon_0 = 0$.

Example 5 (Dilations Group)

$$\tilde{x} = \varepsilon x,$$
$$\tilde{y} = \varepsilon^2 y,$$

where $\varepsilon > 0$. The restriction on ε is due to the fact that $\varepsilon = 0$ has no inverse. The identity element corresponds to $\varepsilon_0 = 1$ and the composition law is $\psi(\varepsilon,\delta) = \varepsilon\delta$. This dilation (scaling) group forms a one-parameter Lie group which can also be re-parametrized by $\varepsilon = 1+\alpha$ as

$$\begin{aligned} \tilde{x} &= (1+\alpha)x, \\ \tilde{y} &= (1+\alpha)^2 y,\ \alpha > -1 \end{aligned} \tag{2.3}$$

with $\alpha_0 = 0$, $\psi(\alpha,\beta) = \alpha+\beta+\alpha\beta$ and the inverse element is $\alpha^{-1} = \frac{-\alpha}{1+\alpha}$.

2.1.3 Invariance of Functions under Lie Groups

A mathematical relationship between variables is said to have a symmetry property if these variables can be subjected to a group of transformations and the resulting expression reads the same in the new variables as the original expression. Such relationship is said to be invariant under the transformation group.
The concept of invariant functions is the main point of development of symmetry analysis. The following definition and example illustrate this concept under one-parameter Lie Group of transformation.

Definition 3 *An infinitely differentiable function* $\Phi(\mathbf{x})$ *is said to be invariant under the one-parameter Lie Group of transformation*

$$\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon) \tag{2.4}$$

if and only if for any group transformation

$$\Phi(\tilde{\mathbf{x}}) \equiv \Phi(\mathbf{x}). \tag{2.5}$$

This means that for invariance, the parameter ε *should vanish after the transformation so that the function reads the same in the new variables.*

Example 6 (Invariance of a Parabola under Dilations)

Consider the parabola

$$\Phi(x,y) = \frac{y}{x^2}\,, \quad x \neq 0 \tag{2.6}$$

and the Lie group of dilation given in Example 5 which can be rewritten as

$$\begin{aligned} \tilde{x} &= e^{\rho} x, \\ \tilde{y} &= e^{2\rho} y \end{aligned} \tag{2.7}$$

in which the parameter ρ takes on the whole range of values from $-\infty$ to $+\infty$ and the identity element is $\rho = 0$, (see Example 7).
Using (2.7) to transform (2.6), we get

$$\Phi(\tilde{x}, \tilde{y}) = \frac{\tilde{y}}{\tilde{x}^2} = \frac{y}{x^2} = \Phi(x, y) \ .$$

This means that the function Φ is invariant under (2.7)

2.2 Infinitesimal Form of Lie Groups

In Section 2.1, we defined a one-parameter Lie groups of the form

$$\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon) \tag{2.8}$$

where ε is the group parameter, which without loss of generality, will be assumed to be defined in such a way that the identity element $\varepsilon_0 = 0$. Therefore

$$\mathbf{x} = F(\mathbf{x}; 0). \tag{2.9}$$

2.2.1 Infinitesimal Transformations and Generators

Definition 4 (Infinitesimal Transformation)

Given a one parameter Lie group of transformation (2.8), we expand $\tilde{\mathbf{x}} = F(\mathbf{x};\varepsilon)$ *into its Taylor series in the parameter* ε *in a neighborhood of* $\varepsilon = 0$. *Then, making use of the fact (2.9), we obtain what is called the infinitesimal transformations of the Lie group of transformation (2.8) :*

$$\tilde{\mathbf{x}} = \mathbf{x} + \varepsilon\xi(\mathbf{x}) + \dots \tag{2.10}$$

where

$$\xi(\mathbf{x}) = \left.\frac{\partial\tilde{\mathbf{x}}}{\partial\varepsilon}\right|_{\varepsilon=0}. \tag{2.11}$$

The components of the vector $\xi(\mathbf{x}) = (\xi_1(\mathbf{x}), \xi_2(\mathbf{x}), ..., \xi_n(\mathbf{x}))$ *are called the* infinitesimals *of (2.8).*

We can get the transformation (2.8) from the infinitesimals $\xi(\mathbf{x})$ with the initial conditions $\tilde{\mathbf{x}} = \mathbf{x}$ at $\varepsilon = 0$ by using the following theorem:

Theorem 1 [First Fundamental Theorem of Lie] [5]

There exists a parametrization $\rho(\varepsilon)$ *such that the Lie group of transformations (2.8) is equivalent to the solution of an initial value problem for the system of first-order ODEs given by*

$$\frac{d\tilde{\mathbf{x}}}{d\rho} = \xi(\tilde{\mathbf{x}}), \tag{2.12}$$

with $\qquad \tilde{\mathbf{x}} = \mathbf{x} \quad \textit{when} \quad \rho = 0\,.$

In particular,

$$\rho(\varepsilon) = \int_0^{\varepsilon} \Gamma(\acute{\varepsilon}) d\acute{\varepsilon}, \tag{2.13}$$

$$\Gamma(\varepsilon) = \left. \frac{\partial \psi(a,b)}{\partial b} \right|_{(a,b)=(\varepsilon^{-1},\varepsilon)}, \tag{2.14}$$

and

$$\Gamma(0) = 1. \tag{2.15}$$

Example 7 For the group of dilations (2.3) in Example 5, the law of composition is given by $\psi(\alpha,\beta) = \alpha + \beta + \alpha\beta$ and $\varepsilon^{-1} = -\frac{\varepsilon}{1+\varepsilon}$.
Hence $\frac{\partial \psi(\alpha,\beta)}{\partial \beta} = 1 + \alpha$,

$$\Gamma(\varepsilon) = \left. \frac{\partial \psi(\alpha,\beta)}{\partial \beta} \right|_{(\alpha,\beta)=(\varepsilon^{-1},\varepsilon)} = 1 + \varepsilon^{-1} = \frac{1}{1+\varepsilon} \ , \ \ \Gamma(0) = 1 \ .$$

and

$$\rho = \int_0^{\varepsilon} \Gamma(\acute{\varepsilon}) d\acute{\varepsilon} = \int_0^{\varepsilon} \frac{1}{1+\acute{\varepsilon}} d\acute{\varepsilon} = log(1+\varepsilon) \ .$$

Let $\mathbf{x} = (x,y)$. Then the group

$$\begin{aligned} \tilde{x} &= (1+\varepsilon)x, \\ \tilde{y} &= (1+\varepsilon)^2 y, \ \varepsilon > -1 \end{aligned} \tag{2.16}$$

becomes

$$\tilde{\mathbf{x}} = ((1+\varepsilon)x, (1+\varepsilon)^2 y) \ .$$

Thus $\frac{\partial \tilde{\mathbf{x}}}{\partial \varepsilon} = (x, 2(1+\varepsilon)y)$, and $\xi(\mathbf{x}) = \left. \frac{\partial \tilde{\mathbf{x}}}{\partial \varepsilon} \right|_{\varepsilon=0} = (x, 2y)$.

As a result, (2.12) here becomes

$$\frac{d\tilde{x}}{d\varepsilon} = \frac{\tilde{x}}{1+\varepsilon}\ , \quad \frac{d\tilde{y}}{d\varepsilon} = \frac{2\tilde{y}}{1+\varepsilon}\ . \tag{2.17}$$

with $\tilde{x} = x$, $\tilde{y} = y$ at $\varepsilon = 0$

The solution of the initial value problem (2.17) is given by (2.16). In terms of the parameterization ρ, the group (2.16) becomes

$$\begin{aligned} \tilde{x} &= e^{\rho} x\ , \\ \tilde{y} &= e^{2\rho} y, \quad -\infty < \rho < \infty, \end{aligned} \tag{2.18}$$

with the law of composition $\psi(\rho_1, \rho_2) = \rho_1 + \rho_2$.

Definition 5 (Infinitesimal generator)

The operator

$$\mathscr{X} = \mathscr{X}(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x^i} \tag{2.19}$$

is called the infinitesimal generator (operator) of the one-parameter Lie group of transformations (2.8), where $\mathbf{x} = (x^1, x^2, ..., x^n) \in \mathbb{R}^n$ *and* $\xi(\mathbf{x}) = (\xi_1, \xi_2, ..., \xi_n)$ *are the infinitesimals of (2.8).* As a special case, take $\mathbf{x} = (x, y) \in \mathbb{R}^2$ then,

$$\mathscr{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y},$$

where $\xi(x,y) = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0}$ and $\eta(x,y) = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0}$.

Example 8 For the rotation group (Example 4)

$$\tilde{x} = x\cos\varepsilon - y\sin\varepsilon, \\ \tilde{y} = x\sin\varepsilon + y\cos\varepsilon \tag{2.20}$$

The infinitesimals are given by $\xi(x,y) = \frac{\partial \tilde{x}}{\partial \varepsilon}\Big|_{\varepsilon=0} = -y$ and $\eta(x,y) = \frac{\partial \tilde{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} = x$. The infinitesimal generator is

$$\mathscr{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ . \tag{2.21}$$

Example 9 For the group of translations (Example 3) with a, b are not both zero, the infinitesimal generator is

$$\mathscr{X} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\ . \tag{2.22}$$

2.2.2 Reconstruction of a Transformations Group from its Infinitesimals

The inverse problem is to find the one-parameter Lie group of transformations from its infinitesimal transformations or when the infinitesimal generator is given. This is illustrated through the following example:

Example 10 Let

$$\mathscr{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.23}$$

We ask the question: which group of transformations corresponds to (2.23)? According to the definition of an infinitesimal generator we have

$$\xi(x,y) = x = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0},\ \eta(x,y) = y = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0}$$

with $\tilde{x} = x,\ \tilde{y} = y$ at $\varepsilon = 0$. The solution of this initial value problem as in Example 7 leads to the group of dilations

$$\tilde{x} = e^{\rho} x,\ \ \tilde{y} = e^{\rho} y\ .$$

Remark: The process of determining a one-parameter Lie group of transformation from its infinitesimal transformation (via Theorem 1) is not always an easy task, as the transformations some times can be complicated. However, if we can achieve our objective only using the infinitesimal generators (which fortunately always are linear operators), this will be nice and it is exactly what the following theorem shows:

Theorem 2 [5] *The one-parameter Lie group of transformations (2.8) is equivalent to*

$$\begin{aligned} \tilde{\mathbf{x}} &= e^{\varepsilon \mathscr{X}} \mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \mathscr{X}^k \mathbf{x}, \end{aligned} \tag{2.24}$$

where $\mathscr{X}$ is the infinitesimal generator of the one-parameter Lie transformations group given by (2.19) and the operator $\mathscr{X}^k = \mathscr{X}^k(\mathbf{x})$ is given by $\mathscr{X}^k = \mathscr{X}\mathscr{X}^{k-1}$. In particular, $\mathscr{X}^k \Phi(\mathbf{x})$ is the function obtained by applying the operator $\mathscr{X}$ to the *infinitely differentiable function* $\mathscr{X}^{k-1}\Phi(\mathbf{x})$ *with* $\mathscr{X}^0\Phi(\mathbf{x}) = \Phi(\mathbf{x})$.

The equation (2.24) is called *the exponential map* and the series in it is called *Lie series*.

Example 11 Consider the infinitesimal generator in Example 8

$$\mathscr{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{2.25}$$

corresponding to the rotation group in the x-y plane. We can determine the transformation corresponding to this generator with help of Theorem 2 as follows:

$$\begin{aligned}
\tilde{x} &= e^{\varepsilon\mathscr{X}}x \\
&= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathscr{X}^k x = \mathscr{X}^0 x + \varepsilon\mathscr{X}x + \frac{\varepsilon^2}{2}\mathscr{X}^2 x + \frac{\varepsilon^3}{3!}\mathscr{X}^3 x + \cdots \\
&= x + \varepsilon(-y) + \frac{\varepsilon^2}{2}\mathscr{X}(-y) + \frac{\varepsilon^3}{3!}\mathscr{X}(-x) + \frac{\varepsilon^4}{4!}\mathscr{X}(y) + \frac{\varepsilon^5}{5!}\mathscr{X}(x) + \cdots \\
&= (1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \cdots)x - (\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \cdots)y \\
&= x\cos\varepsilon - y\sin\varepsilon, \qquad (2.26) \\
\tilde{y} &= e^{\varepsilon\mathscr{X}}y \\
&= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathscr{X}^k y = \mathscr{X}^0 y + \varepsilon\mathscr{X}y + \frac{\varepsilon^2}{2!}\mathscr{X}^2 y + \frac{\varepsilon^3}{3!}\mathscr{X}^3 y + \cdots \\
&= y + \varepsilon x + \frac{\varepsilon^2}{2!}\mathscr{X}x + \frac{\varepsilon^3}{3!}\mathscr{X}(-y) + \cdots \\
&= (\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \cdots)x + (1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \cdots)y \\
&= x\sin\varepsilon + y\cos\varepsilon. \qquad (2.27)
\end{aligned}$$

Remark 1 In Subsection 2.1.3, we discussed the invariance of functions under lie groups of transformations. The following theorem provides an interesting criterion of invariance (symmetry) of functions (also of differential equations which are treated as differential functions of the independent and dependent variables and their derivatives) using the infinitesimal generator. In this thesis, we have used the word infinitesimal generator with symmetry generator interchangeably.

Theorem 3 [5] *An infinitely differentiable function* $\Phi(\mathbf{x})$ *is an invariant of the Lie group of transformations (2.8);* $\tilde{\mathbf{x}} = \mathbf{F}(\mathbf{x}\,;\varepsilon)$ *if and only if*

$$\mathscr{X}\,\Phi(\mathbf{x}) \equiv 0, \tag{2.28}$$

where $\mathscr{X}$ *is the infinitesimal generator of the symmetry transformations.*

2.3 Extensions of Transformations (Prolongations)

The main aim of studying Lie groups of transformations and their generators is to use them in analyzing and solving DEs. However, from the previous sections it is clear that the transformations that we have discussed do not include partial derivatives, so it is necessary to define an extension (prolongation) of the infinitesimal transformations and their generators to include the transformations of derivatives. To accomplish this, we use an algorithmic procedure for extension of the infinitesimal transformations separately according to the numbers of independent and dependent variables.

2.3.1 Prolongations in the Plane

Consider the one-parameter Lie group of point transformations

$$\begin{aligned} \tilde{x} &= F(x,y;\varepsilon), \\ \tilde{y} &= G(x,y;\varepsilon), \end{aligned} \tag{2.29}$$

acting on the $(x,y)-$space and consider the kth-order differential equation

$$E(x,y,y_1,...,y_k) = 0, \tag{2.30}$$

where $y = y(x)$ and $y_k = y^{(k)} = \frac{d^k y}{dx^k}$, with $k \geq 1$. We extend (2.29) to $(x,y,y_1,...,y_k)$-space by requiring that (2.29) preserves the kth order contact conditions relating the differentials $dx, dy, dy_1, ..., dy_k$:

$$\begin{aligned} dy &= y_1 dx, \\ &\vdots \\ dy_k &= y_{k+1} dx, \end{aligned} \tag{2.31}$$

The above also holds for the target coordinates $(\tilde{x}, \tilde{y})$.

Lemma 1 [8] *The kth order contact condition $dy_k - y_{k+1}dx = 0$ is preserved by the Lie group of point transformations (2.29). So to all orders of derivatives,*

$$\begin{aligned} d\tilde{y} - \tilde{y}_1 d\tilde{x} &= dy - y_1 dx, \\ d\tilde{y_1} - \tilde{y_2} d\tilde{x} &= dy_1 - y_2 dx, \\ &\vdots \\ d\tilde{y}_k - \tilde{y}_{k+1} d\tilde{x} &= dy_k - y_{k+1} dx. \end{aligned} \tag{2.32}$$

The kth extended group of point transformations are given by the following theorem which also refers to the very important point that assures that the transformation including derivatives is invertible; that is, the group of kth extended point transformations is a Lie group:

Theorem 4 [6] *The kth prolongation of the one-parameter Lie group of point transformations (2.29) is the following one-parameter Lie group of point transformations acting on $(x, y, y_1, ..., y_k)$-space:*

$$\begin{aligned} &\tilde{x} = F(x, y; \varepsilon), \\ &\tilde{y} = G(x, y; \varepsilon), \\ &\tilde{y_1} = G_1(x, y, y_1; \varepsilon) = (\tfrac{\partial G}{\partial x} + y_1 \tfrac{\partial G}{\partial y})/(\tfrac{\partial F}{\partial x} + y_1 \tfrac{\partial F}{\partial y}), \\ &\tilde{y_2} = G_2(x, y, y_1, y_2; \varepsilon) = (\tfrac{\partial G_1}{\partial x} + y_1 \tfrac{\partial G_1}{\partial y} + y_2 \tfrac{\partial G_1}{\partial y_1})/(\tfrac{\partial F}{\partial x} + y_1 \tfrac{\partial F}{\partial y}), \\ &\qquad \vdots \\ &\tilde{y_k} = G_k(x, y, y_1, ..., y_k; \varepsilon) \\ &\quad = (\tfrac{\partial G_{k-1}}{\partial x} + y_1 \tfrac{\partial G_{k-1}}{\partial y} + \cdots + y_k \tfrac{\partial G_{k-1}}{\partial y_{k-1}})/(\tfrac{\partial F}{\partial x} + y_1 \tfrac{\partial F}{\partial y})\ . \end{aligned} \tag{2.33}$$

Remark 2 In terms of the total differentiation operator with respect to x defined by

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \cdots + y_{n+1} \frac{\partial}{\partial y_n} + \cdots\,, \tag{2.34}$$

the kth prolonged Lie group of point transformations (2.38) is given by

$$\begin{aligned} \tilde{x} &= F(x,y;\varepsilon), \\ \tilde{y} &= G(x,y;\varepsilon), \\ \tilde{y}_i &= G_i(x,y,y_1,...,y_i;\varepsilon) = \tfrac{DG_{i-1}(x,y,y_1,...,y_{i-1};\varepsilon)}{DF(x,y;\varepsilon)} , \end{aligned} \tag{2.35}$$

with $G_0 = G(x,y;\varepsilon)$, $i = 1,2,...,k$.

Remark 3 From Section 2.2, we know that the one-parameter Lie group of point transformations can be constructed from its infinitesimal generator. Since this generator is the best tool to test invariance of functions, then we should have the kth prolongation form of the infinitesimal generator of Lie groups of point transformations.

The infinitesimal form of (2.29) acting on the $(x,y)-$space is given by

$$\begin{aligned} \tilde{x} &= F(x,y;\varepsilon) = x + \varepsilon\xi(x,y), \\ \tilde{y} &= G(x,y;\varepsilon) = y + \varepsilon\eta(x,y), \end{aligned} \tag{2.36}$$

with the infinitesimals $\xi(x,y), \eta(x,y)$ and the corresponding infinitesimal symmetry generator

$$\mathscr{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} . \tag{2.37}$$

The kth order prolongation of (2.36), given by

$$
\begin{aligned}
\tilde{x} &= F(x,y;\varepsilon) = x + \varepsilon\xi(x,y),\\
\tilde{y} &= G(x,y;\varepsilon) = y + \varepsilon\eta(x,y),\\
\tilde{y_1} &= G_1(x,y,y_1;\varepsilon) = y_1 + \varepsilon\eta^{(1)}(x,y,y_1),\\
&\vdots\\
\tilde{y_k} &= G_k(x,y,y_1,...,y_k;\varepsilon) = y_k + \varepsilon\eta^{(k)}(x,y,y_1,...,y_k)
\end{aligned}
\tag{2.38}
$$

has the kth-prolonged infinitesimals $\xi(x,y),\eta(x,y),\eta^{(1)}(x,y,y_1),...,\eta^{(k)}(x,y,y_1,...,y_k)$, with the associated kth-prolonged symmetry generator

$$
\mathscr{X}^{(1)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y_1} + \cdots + \eta^{(k)}\frac{\partial}{\partial y_k}\ ,k = 1,2,... \tag{2.39}
$$

What remains is to determine the additional coefficients $\eta^{(1)},\ldots,\eta^{(k)}$.

Theorem 5 [5] *The prolonged infinitesimals* $\eta^{(k)}$ *satisfy the recursion formula*

$$
\eta^{(k)}(x,y,y_1,...,y_k) = D\eta^{(k-1)}(x,y,y_1,...,y_{k-1}) - y_k D\xi\ ,k = 1,2,... \tag{2.40}
$$

with $\eta^{(0)} = \eta(x,y)$.

Corollary 1 [6]

1. $\eta^{(k)}$ *is linear in* y_k *for* $k \geq 1$.
2. $\eta^{(k)}$ *is is a polynomial in* $y_i,\ i = 1,2,...,k$, *with linear homogeneous coefficients in* $\xi(x,y),\eta(x,y)$ *and their partial derivatives up to kth-order.*

2.3.2 Prolongations with m Dependent and n Independent Variables

The one-parameter Lie group of infinitesimal transformations acting on the (x,u)-space, [$u = (u^1, u^2, ..., u^m)$ is called the dependent variable (is a function of the independent variable $x = (x^1, x^2, ..., x^n)$)] is given by [12]

$$\begin{aligned} \tilde{x}^i &= F^i(x,u;\varepsilon) = x^i + \varepsilon \xi^i(x,u),\ i = 1,2,\ldots,n \\ \tilde{u}^\alpha &= G^\alpha(x,u;\varepsilon) = u^\alpha + \varepsilon \eta^\alpha(x,u),\ \alpha = 1,2,\ldots,m \end{aligned} \tag{2.41}$$

Consider the kth-order differential equation

$$E^\alpha(x,u,u_{(1)},...,u_{(k)}) = 0, \tag{2.42}$$

where $u_{(1)},...,u_{(k)}$ denote the collections of all first, second,..., kth-order derivatives, that is $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\},...,u_{(k)} = \{u_{i_1 i_2 \ldots i_k}^\alpha\}$ for $\alpha = 1,2,...,m$ and $i, j, i_1, i_2,...,i_k = 1,2,...,n$. The corresponding symmetry generator is

$$\mathscr{X} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\ . \tag{2.43}$$

The kth prolongation of (2.41) is the following Lie group of transformations acting on the $(x,u,u_{(1)},...,u_{(k)})$-space [5]:

$$
\begin{aligned}
\tilde{x}^i &= x^i + \varepsilon \xi^i(x,u),\\
\tilde{u}^\alpha &= u^\alpha + \varepsilon \eta^\alpha(x,u),\\
\tilde{u_i}^\alpha &= u_i^\alpha + \varepsilon \eta_i^{(1)\alpha}(x,u,u_{(1)}),\\
\tilde{u_{ij}}^\alpha &= u_{ij}^\alpha + \varepsilon \eta_{ij}^{(2)\alpha}(x,u,u_{(1)},u_{(2)}),\\
&\vdots\\
\tilde{u}^\alpha_{i_1 i_2 \ldots i_k} &= u^\alpha_{i_1 i_2 \ldots i_k} + \varepsilon \eta^{(k)\alpha}_{i_1 i_2 \ldots i_k}(x,u,u_{(1)},\ldots,u_{(k)})\ ,
\end{aligned}
\tag{2.44}
$$

where the prolonged infinitesimals $\eta_i^{(1)\alpha}, \eta_{ij}^{(2)\alpha}, \cdots, \eta^{(k)\alpha}_{i_1 i_2 \ldots i_k}$ are given by the following theorem:

Theorem 6 [5] *The prolonged infinitesimals in (2.44) are given, recursively by the prolongation formulas*

$$
\begin{aligned}
\eta_i^{(1)\alpha} &= D_i \eta^\alpha - u_j^\alpha D_i \xi^j,\\
\eta_{ij}^{(2)\alpha} &= D_j \eta_i^{(1)\alpha} - u_{il}^\alpha D_j \xi^l,\\
&\vdots\\
\eta^{(k)\alpha}_{i_1 i_2 \ldots i_k} &= D_{i_k} \eta^{(k-1)\alpha}_{i_1 i_2 \ldots i_{k-1}} - u^\alpha_{i_1 i_2 \ldots i_{k-1} j} D_j \xi^j\ ,
\end{aligned}
\tag{2.45}
$$

where $i_q = 1,2,\ldots,n$ for $q = 1,2,\ldots,k$ with $k \geq 2$ and the total differentiation operator with respect to x^i is given (with assuming summation over a repeated index) by

$$
D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u^\alpha_{i i_1 i_2 \ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_k}}\ . \tag{2.46}
$$

with $$u^\alpha_{i_1 i_2 \ldots i_k} = \frac{\partial^k u^\alpha}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}\ ,\ u^\alpha_{i i_1 i_2 \ldots i_k} = \frac{\partial^{k+1} u^\alpha}{\partial x_i \partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}.$$

The kth prolonged symmetry generator of (2.43) will be

$$\mathscr{X}^{(k)} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \cdots + \eta_{i_1 i_2 \dots i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha}\ ,\ k \geq 1. \tag{2.47}$$

2.4 Multi-parameter Lie Group of Transformations

Definition 6 *Let the vector* $\mathbf{x} = (x^1, x^2, \dots, x^n)$ *lie in a region* $\mathscr{D} \subset \mathbb{R}^n$*. For each* $\mathbf{x}$ *and* $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ *in a set* $\mathscr{S} \subset \mathbb{R}^r$*, Define the set of one-to-one transformations onto* $\mathscr{D}$

$$\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon), \tag{2.48}$$

which is said to be an r-parameter group of transformations G_r *with respect to the binary operation of composition* $\psi(\varepsilon, \delta) = (\psi_1(\varepsilon, \delta), \psi_2(\varepsilon, \delta), \dots, \psi_r(\varepsilon, \delta))$,
$\delta = (\delta_1, \delta_2, \dots, \delta_r)$ *if the following conditions are satisfied:*
(i) $(\mathscr{S}, \psi)$ *is a group.*
(ii) there is an identity element $\varepsilon = \varepsilon_0 \in \mathscr{S}$ *such that*

$$\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon_0) = \mathbf{x}$$

(iii) (invertibility property) If we compose the two members of the group

$$\tilde{\mathbf{x}} = F(\mathbf{x}; \varepsilon_1)\ , \tilde{\tilde{\mathbf{x}}} = F(\tilde{\mathbf{x}}; \varepsilon_2)$$

the result is a transformation that is a member of the group

$$\tilde{\tilde{\mathbf{x}}} = F(F(\mathbf{x};\varepsilon_1);\varepsilon_2) = F(\mathbf{x};\varepsilon_3), \;\; \varepsilon_3 \in \mathscr{S},$$

where $\varepsilon_3 = \psi(\varepsilon_1, \varepsilon_2)$

Definition 7 (*r*-parameter Lie group of transformations)

An r-parameter Group G_r of transformations

$$\tilde{\mathbf{x}} = F(\mathbf{x};\varepsilon), \tag{2.49}$$

with $x = (x_1, x_2, \dots, x_n)$ and parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ is called an r-parameter Lie group of transformations if

1. *ε is a continuous parameter, without loss of generality, with the identity element* $\varepsilon = 0$.

2. *F is an infinitely differentiable function with respect to $\mathbf{x}$ in $\mathscr{D}$ and an analytic function of ε in $\mathscr{S}$.*

3. *The composition law for parameters denoted by $\psi(\varepsilon, \delta)$ is an analytic function of $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$ and $\delta = (\delta_1, \delta_2, \dots, \delta_r)$ in $\mathscr{S}$.*

Definition 8 *The infinitesimal generator $\mathscr{X}_a$, corresponding to the parameter ε_a of the r-parameter Lie group of transformations (2.48) is given by*

$$\mathscr{X}_a = \sum_{i=1}^{n} \xi_a^i(\mathbf{x}) \frac{\partial}{\partial x_i}, \tag{2.50}$$

where $\xi_a^i = \frac{\partial \tilde{x}_i}{\partial \varepsilon_a}\Big|_{\varepsilon_a=0}$, $a = 1, \ldots, r$, $i = 1, \ldots, n$.

2.5 Lie Algebras

As we already know, each parameter of an r-parameter Lie group of transformations leads to a symmetry generator (which is a linear operator). These symmetry generators belong to an r-dimensional linear vector space in which any linear combination of generators is also linear operator and the way of ordering generators is not important, that is, the symmetry group of transformation commute and this leads to the additional structure in the mentioned vector space called the commutator. Before going to the precise definition of this special vector space (known as an r-dimensional Lie Algebra), we give some related concepts and results.

2.5.1 The Commutators

Definition 9 *Let G_r be the r-parameter Lie group of transformations (2.49) with the symmetry generators $\mathscr{X}_a$,$a = 1, 2, ..., r$ given by (2.50). The commutator (Lie bracket)* [,] *of two symmetry generators $\mathscr{X}_a, \mathscr{X}_b$ is the first order operator generated as follows*

$$[\mathscr{X}_a, \mathscr{X}_b] = \mathscr{X}_a \mathscr{X}_b - \mathscr{X}_b \mathscr{X}_a \tag{2.51}$$

Theorem 7 [6] *The commutator $[\mathscr{X}_a, \mathscr{X}_b]$ of any two symmetry generators $\mathscr{X}_a, \mathscr{X}_b$ of an r-parameter Lie group of transformation is also a symmetry generator and can* *be written as*

$$[\mathscr{X}_a, \mathscr{X}_b] = \lambda_{ab}^{c} \mathscr{X}_c, \ \ (\textit{sum over } c = 1,..,r) \ \forall \ \ a,b = 1,..,r \tag{2.52}$$

where the coefficients λ_{ab}^{c} *are constants.*

Definition 10 *The equations (2.52) are called the commutation relations of the r-parameter Lie group of transformations (2.49) with the symmetry generators (2.50), and the constants* λ_{ab}^{c} *are called structure constants.*
By direct computation, one can get the following properties:

Corollary 2 [8]

1. *(Anti-symmetry property) The commutator is anti-symmetric, i.e. for any two symmetry generators* $\mathscr{X}_a, \mathscr{X}_b$*,*

$$[\mathscr{X}_a, \mathscr{X}_b] = -[\mathscr{X}_b, \mathscr{X}_a] \ . \tag{2.53}$$

2. (*Jacobi identity) For any three symmetry generators* $\mathscr{X}_a, \mathscr{X}_b, \mathscr{X}_c$*,*

$$[\mathscr{X}_a, [\mathscr{X}_b, \mathscr{X}_c]] + [\mathscr{X}_b, [\mathscr{X}_c, \mathscr{X}_a]] + [\mathscr{X}_c, [\mathscr{X}_a, \mathscr{X}_b]] = 0 \ . \tag{2.54}$$

3. *The structure constants defined by the commutation relations (2.52) satisfy the following relations:*

(a) *The structure constants are anti-symmetric in the lower indices,*

$$\lambda_{ab}^{c} = -\lambda_{ba}^{c} \tag{2.55}$$

(b) *(Lie identity) It follows from the Jacobi identity,*

$$\lambda_{ab}^{l}\lambda_{lc}^{\mu} + \lambda_{bc}^{l}\lambda_{la}^{\mu} + \lambda_{ca}^{l}\lambda_{lb}^{\mu} = 0 \ (sum\ over\ l = 1, 2, ..., r). \tag{2.56}$$

Definition 11 (Lie algebra)

A Lie algebra $\mathscr{L}$ is a vector space over a field $\mathscr{F}$ with a given bilinear commutation law (the commutator) satisfying the properties (2.52) - (2.54).

In particular, the Lie group of transformations G_r (whose set of symmetry generators is $\{\mathscr{X}_a\}, a = 1, 2, ..., r$), forms an r-dimensional Lie algebra over $\mathbb{R}$ denoted by $\mathscr{L}_r$, and its basis is the symmetry generators $\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_r$.

2.5.2 The Commutators Table

In applications, it is convenient to summarize a Lie algebra by setting up the table of commutators of its basis. To illustrate the idea, consider the following example:

Example 12 Let $\mathscr{L}_6$ be the vector space with the following basis [17]

$$\mathscr{X}_1 = \frac{\partial}{\partial x}\ ,\ \mathscr{X}_2 = \frac{\partial}{\partial y}\ ,\ \mathscr{X}_3 = \frac{\partial}{\partial u}\ ,\ \mathscr{X}_4 = y\frac{\partial}{\partial y}\ ,$$

$$\mathscr{X}_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}\ ,\ \mathscr{X}_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}\ .$$

It is easy to calculate, by the definition, the commutators $[\mathscr{X}_a, \mathscr{X}_b],\ a,b = 1,2,...,6,$ and verify that they satisfy the properties of a Lie algebra $\mathscr{L}_6$ and the table of the commutators is as shown in the following table :

$[\mathscr{X}_a, \mathscr{X}_b]$	$\mathscr{X}_1$	$\mathscr{X}_2$	$\mathscr{X}_3$	$\mathscr{X}_4$	$\mathscr{X}_5$	$\mathscr{X}_6$
$\mathscr{X}_1$	0	0	0	0	$\mathscr{X}_1$	0
$\mathscr{X}_2$	0	0	0	$\mathscr{X}_3$	0	$\mathscr{X}_2$
$\mathscr{X}_3$	0	0	0	0	$3\mathscr{X}_3$	$-2\mathscr{X}_3$
$\mathscr{X}_4$	0	$-\mathscr{X}_3$	0	0	$3\mathscr{X}_4$	$-3\mathscr{X}_4$
$\mathscr{X}_5$	$-\mathscr{X}_1$	0	$-3\mathscr{X}_3$	$-3\mathscr{X}_4$	0	0
$\mathscr{X}_6$	0	$-\mathscr{X}_2$	$2\mathscr{X}_3$	$3\mathscr{X}_4$	0	0

Table 2.1: Commutators table

Note that the table is anti-symmetric with zeros on the main diagonal which shows the definition of the commutator and its properties.

Chapter 3

Noether Symmetries and Conservation Laws

Apart from Lie point symmetries there are some interesting symmetries that can be associated with differential equations DEs which possess a Lagrangian. These symmetries are called Noether symmetries and describe physical features of the DEs in terms of conservation laws they admit. For more knowledge on the basic theory and some advanced concepts of Noether symmetries and conservation laws of DEs, one can refer to Noether [13], Olver [25], Bluman and Kumei [6], Cantwell [8], Stephani [26], Ibragimov [17, 18, 20, 21], Bluman, Cheviakov, and Anco [7] and Kosmann-Schwarzbach [23], [22].

The *killing vectors* defined via *Lie derivative* of the metric being zero give conservation laws admitted by the Lorentzian metrics and space-time metrics can be used to construct Lagrangians, *Euler-Lagrange equations* and consequently Noether symmetries from the point of view of DEs via Noether Theorem.

The main objective of this chapter is to present without proof a brief discussion of some principal concepts and some theorems which are of essential use and related to Noether's method for constructing conservation laws for systems of DEs. We present briefly in Section 3.3 the concepts of Lorentzain metrics, Killing vectors and the connections between them and Noether symmetries.

In Section 3.4, we revisit the work of Feroze [1] on Noether symmetries of the Euler Lagrange equations of a Lorentzian metric, known as Bertotti - Robinson metric and re-produce her work to show the procedure adopted there.

3.1 Conservation Laws

Consider a system of partial differential equations

$$F^k(x,u,u_{(1)},\dots,u_{(r)}) = 0, \quad k = 1,\dots,m \tag{3.1}$$

where $x = (x^1,x^2,\dots,x^n)$ denotes the n independent variables, $u = (u^1,u^2,\dots,u^m)$ denotes the m dependent variables, and $u_{(l)}$ denotes the set of coordinates corresponding all lth-order partial derivatives of u with respect to x, $l = 1,2,\dots,r$. Using the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^k \frac{\partial}{\partial u^k} + u_{ij}^k \frac{\partial}{\partial u_j^k} + ... + u_{ii_1 i_2 \ldots i_n}^k \frac{\partial}{\partial u_{ii_1 i_2 \ldots i_n}^k} + \ldots, \quad i, i_1, \ldots, i_n = 1, \ldots, n, \tag{3.2}$$

in which the summation over a repeated index is used, we have

$u_{(1)} = u_i^k = D_i(u^k), u_{(2)} = u_{ij}^k = D_i(u_j^k) = D_i D_j(u^k), \ldots$

Definition 12 (Conservation law)

We say that the system of partial differential equations (3.1) has a conservation law if there exists an n-dimensional vector $T = (T^1, T^2, \ldots, T^n)$ *with* $T^i = T^i(x, u, u_{(1)}, \ldots, u_{(r)})$, $i = 1, 2, \ldots, n$ *satisfying*

$$D_i(T) = 0 \tag{3.3}$$

at any solution of the system (3.1) and then T *is called a conserved vector.*

So by definition, the conservation law of a system is an equation in divergence- free form: div $T = D_i(T^i) = 0$ of the vector function $T^i = T^i(x, u, u_{(1)}, \ldots, u_{(r)})$ which must hold for any $u(x)$ of (3.1).

The Lagrangian of a dynamical system is a function that describes and summarizes the dynamics of the system. In classical mechanics, the Lagrangian is defined as $L \equiv T - V$, where T is the total kinetic energy and V is the total potential energy. Mathematically, we can define the Lagrangian of the system (3.1) as follows:

Definition 13 (Lagrangian of a system)

If there exists a function $L = L(x, u, u_{(1)}, \ldots, u_{(l)})$, $l \leq r$ *such that the system (3.1) can* *be written as* $\frac{\delta}{\delta u^k}(L) = 0$*, where* $\frac{\delta}{\delta u^k}$ *denotes the Euler-Lagrange operators given by*

$$\frac{\delta}{\delta u^k} = \frac{\partial}{\partial u^k} - D_i \frac{\partial}{\partial u_i^k} + D_i D_j \frac{\partial}{\partial u_{ij}^k} + \cdots + (-1)^l D_{i_1} \ldots D_{i_l} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_l}^k}, \quad k = 1, \ldots, m, \tag{3.4}$$

then L is called a Lagrangian of the system (3.1)

Definition 14 (Euler-Lagrange equations)

The differential equations of the form

$$\frac{\delta}{\delta u^k}(L) = 0, \quad k = 1, \ldots, m \tag{3.5}$$

if they exist, are called the Euler-Lagrange equations associated with system (3.1).

Example 13 Consider the Lagrangian

$$L(x, y, y') = \frac{1}{2} y'^2 - \frac{1}{2} y^2. \tag{3.6}$$

The Euler-Lagrange equation for this Lagrangian is given by

$$\frac{\partial L}{\partial y} - D_x \frac{\partial L}{\partial y'} = 0. \tag{3.7}$$

By a direct substitution of the Lagrangian (3.6), we find that the EulerLagrange equation represents the second order differential equation

$$y'' + y = 0 \tag{3.8}$$

Theorem 8 [6] *For any twice continuously differentiable function*

$H(x,u,u_{(1)},...,u_{(s)})$, we have

$$\frac{\delta}{\delta u^k} D_i H(x,u,u_{(1)},...,u_{(s)}) \equiv 0,\ k = 1,2,...,m;\ i = 1,2,...,n$$

where $\frac{\delta}{\delta u^k}$ and D_i are the Euler-Lagrange and the total differentiation operators given respectively by equations (3.4) and (3.2).

The following two corollaries follow immediately from Theorem 8:

Corollary 3 [6] *The Euler-Lagrange equations for a Lagrangian L are identically zero if L can be expressed in divergence form:*

$$L = D_i H^i(x,u,u_{(1)},...,u_{(s)})\ .$$

Corollary 4 *Two Lagrangians L and L' have the same set of Euler- Lagrange equations if $L - L' = div\ A$ for some vector*

$$A(x,u,u_{(1)},...,u_{(s)}) = (A^1, A^2, ..., A^n)$$

Remark: Corollary 4 refers to the fact that there can be more than one Lagrangian for a given differential equation, for example,

$$L_1 = \tfrac{1}{2}y'^2 \quad \text{and} \quad L_2 = \tfrac{1}{2}yy'^2 - \tfrac{1}{6}xy'^3$$

are associated with the same Euler- Lagrange equation $y'' = 0$, because

$$\frac{\partial L_1}{\partial y} - D_x\frac{\partial L_1}{\partial y'} = \frac{\partial}{\partial y}\frac{1}{2}y'^2 - D_x\frac{\partial}{\partial y'}\frac{1}{2}y'^2 = -D_x y' = -y'' = 0,$$

and similarly

$$\begin{aligned}\frac{\partial L_2}{\partial y} - D_x\frac{\partial L_2}{\partial y'} &= \tfrac{1}{2}y'^2 - D_x yy' + \tfrac{1}{2}D_x(xy'^2)\\ &= \tfrac{1}{2}y'^2 - y'^2 - yy'' + \tfrac{1}{2}y'^2 + xy'y''\\ &= -y''(y-xy') = 0,\end{aligned}$$

which also implies that $y'' = 0$.

Definition 15 (The Action Integral)

Given a Lagrangian $L(x,u,u_{(1)},...,u_{(l)})$ defined on a domain Ω in the space of $x = (x_1,x_2,...,x_n) \in \mathbb{R}^n$, the functional

$$S[u] = \int_\Omega L(x,u,u_{(1)},...,u_{(l)})dx \tag{3.9}$$

is called the action integral of a Lagrangian L.

Definition 16 (Invariance of the Action Integral)

Let G_α be an α-parameter group of transformations

$$\begin{aligned}\tilde{x} &= x + \varepsilon\xi(x,u,u_{(1)},...,u_{(r)}) + O(\varepsilon^2),\\ \tilde{u} &= u + \varepsilon\eta(x,u,u_{(1)},...,u_{(r)}) + O(\varepsilon^2).\end{aligned} \tag{3.10}$$

The functional (3.9) is said to be invariant with respect to the group G_α if for all transformations of the group and all functions $u = u(x)$ the following equality is fulfilled irrespective of the choice of the domain of integration

$$\int_\Omega L(x,u,u_{(1)},...,u_{(r)})dx = \int_{\tilde{\Omega}} L(\tilde{x},\tilde{u},\tilde{u}_{(1)},...,\tilde{u}_{(r)})d\tilde{x}, \tag{3.11}$$

where $\tilde{u}$ and $\tilde{\Omega}$ are the images of u and Ω, respectively, under the group G_α.

3.2 Noether Symmetries

Definition 17 (Lie-Bäcklund generator)

Consider an r-th order differential equation given by (3.1), a Lie-Bäcklund generator is defined by the formula

$$X^{[s]} = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k} + \eta_i^{(1)k} \frac{\partial}{\partial u_i^k} + \eta_{i_1 i_2}^{(2)k} \frac{\partial}{\partial u_{i_1 i_2}^k} + \cdots + \eta_{i_1 \ldots i_s}^{(s)k} \frac{\partial}{\partial u_{j i_1 \ldots i_s}^k} \tag{3.12}$$

which is the s-th prolongation of the generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k} \tag{3.13}$$

where ξ^i, η^k are any differentiable functions of $x, u, u_{(1)}, \ldots, u_{(r)}$ and $\eta_i^{(1)k}, \eta_{i_1 i_2}^{(2)k}, \ldots,$ $\eta_{i_1 \ldots i_s}^{(s)k}$ are determined by the prolongation formulas [5],

$$\begin{aligned}
\eta_i^{(1)k} &= D_i(\eta^k) - u_j^k D_i(\xi^j), \\
\eta_{i_1 i_2}^{(2)k} &= D_{i_2}(\eta_{i_1}^{(1)k}) - u_{j i_1}^k D_{i_2}(\xi^j), \\
&\vdots \\
\eta_{i_1 \ldots i_s}^{(s)k} &= D_{i_s} \eta_{i_1 \ldots i_{s-1}}^{(s-1)k} - u_{j i_1 \ldots i_{s-1}}^k D_{i_s}(\xi^j), \quad s = 1, 2, \ldots
\end{aligned} \tag{3.14}$$

which also can be rewritten using the Lie characteristic functions [29]
$W^k = \eta^k - \xi^j u_j^k$ as

$$\begin{aligned}
\eta_i^{(1)k} &= D_i(W^k) + \xi^j u_{ij}^k, \\
\eta_{i_1 i_2}^{(2)k} &= D_{i_1 i_2}(W^k) + \xi^j u_{j i_1 i_2}^k, \\
&\vdots \\
\eta_{i_1 \ldots i_s}^{(s)k} &= D_{i_1} \ldots D_{i_s}(W^k) + \xi^j u_{j i_1 \ldots i_s}^k, \quad s = 1, 2, \ldots
\end{aligned} \tag{3.15}$$

So, using W^k we have an alternative form of the operator (3.12) as [17]

$$X^{[s]} = \xi^i D_i + W^k \frac{\partial}{\partial u^k} + D_i(W^k) \frac{\partial}{\partial u_i^k} + D_{i_1} D_{i_2}(W^k) \frac{\partial}{\partial u_{i_1 i_2}^k} + \ldots \tag{3.16}$$

Definition 18 (Noether operator)

The operator

$$N^i = \xi^i + W^k \frac{\delta}{\delta u_i^k} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^k) \frac{\delta}{\delta u_{i i_1 i_2 \ldots i_s}^k}, \ i = 1, \ldots, n \tag{3.17}$$

is called the Noether operator associated with the Lie-Bäcklund operator (3.16), where the Euler-Lagrange operators with respect to derivatives of u^k are obtained from (3.4) by replacing u^k by the corresponding derivatives, .e.g

$$\frac{\delta}{\delta u_i^k} = \frac{\partial}{\partial u_i^k} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s}(W^k) \frac{\partial}{\partial u_{i j_1 j_2 \ldots j_s}^k} \ . \tag{3.18}$$

Definition 19 (Noether symmetry)

For a Lie-Bäcklund generator $X^{[s]}$ of the form (3.12), if there exists a vector $B = (B^1, B^2, \ldots, B^n)$ such that

$$X^{[s]} L + L D_i \xi^i = D_i B^i, \ i = 1, 2, \ldots, n \tag{3.19}$$

then, $X^{[s]}$ is called a Noether symmetry associated with the Lagrangian L

Remark: The vector function B in (3.19) is called a gauge function, and if it is zero, then $X^{[s]}$ is called a variational symmetry or a strict Noether symmetry.

Theorem 9 (Noether's Theorem) [21]

Let X be a Noether symmetry associated with the Lagrangian L, then for any X there exists a vector $T = (T^1, T^2, ..., T^n)$ termed by

$$T^i = N^i(L) - B^i, \quad i = 1, ..., n, \tag{3.20}$$

which is a conserved vector of the Euler-Lagrange equations (3.5). i.e. the equations (3.5) admit a conservation law, $D_i(T^i) = 0$.

Example 14 (The free motion of a mass in the absence of body forces) Lagrangian for a mass moving in the absence of any body forces is given by its kinetic energy

$$L = \frac{1}{2}m((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2) \tag{3.21}$$

where " . " denotes differentiation with respect to t . The Euler-Lagrange equations corresponding to (3.21) are as follows

$$\frac{\delta}{\delta x^i}(L) = \frac{\partial L}{\partial x^i} - D_t \frac{\partial L}{\partial \dot{x}^i} = -D_t(m\dot{x}^i) = -m\ddot{x}^i = 0\ , \quad i = 1, 2, 3 \tag{3.22}$$

which states that the acceleration of the particle is zero. The Euler system (3.22) is invariant under the four-parameter group of space-time translations

$$X_1 = \frac{\partial}{\partial x^1}\ ,\quad X_2 = \frac{\partial}{\partial x^2}\ ,\quad X_3 = \frac{\partial}{\partial x^3}\ ,\quad X_4 = \frac{\partial}{\partial t} \tag{3.23}$$

and the three-parameter group of rotations [8]

$$X_5 = x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\ ,\quad X_6 = x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}\ ,\quad X_7 = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\ . \tag{3.24}$$

Using (3.23) and (3.24), conserved vectors corresponding to each of the groups can be constructed. In this case, where time is the only independent variable, the conserved vectors have only one component. Conserved vectors corresponding to each of the independent space translations X_1, X_2, X_3 are the three components of the linear momentum,

$$\begin{aligned} P^1 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = \tfrac{\partial L}{\partial \dot{x}^1} = m\dot{x}^1, \\ P^2 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = \tfrac{\partial L}{\partial \dot{x}^2} = m\dot{x}^2, \\ P^3 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = \tfrac{\partial L}{\partial \dot{x}^3} = m\dot{x}^3. \end{aligned} \tag{3.25}$$

The vector momentum is usually denoted **P**, and so the conserved quantity is

$$\mathbf{P} = m\mathbf{v}. \tag{3.26}$$

The conserved vector corresponding to in variance under time translation X_4 is the kinetic energy of the particle,

$$\begin{aligned} -E &= \xi L + (\eta^i - \xi \dot{x}^i)\frac{\partial L}{\partial \dot{x}^i} = L - \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} \\ &= \frac{1}{2}m((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2) - m((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2) \\ &= -\frac{1}{2}m((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2), \end{aligned} \tag{3.27}$$

or

$$E = \frac{1}{2}mv^2. \tag{3.28}$$

Invariance under rotations X_5, X_6, X_7 gives the three components of angular momentum,

$$\begin{aligned} M^1 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = -x^3 \tfrac{\partial L}{\partial \dot{x}^2} + x^2 \tfrac{\partial L}{\partial \dot{x}^3} = m(x^2\dot{x}^3 - x^3\dot{x}^2), \\ M^2 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = \ x^3 \tfrac{\partial L}{\partial \dot{x}^1} - x^1 \tfrac{\partial L}{\partial \dot{x}^3} \ = m(x^3\dot{x}^1 - x^1\dot{x}^3), \\ M^3 &= \xi L + (\eta^i - \xi \dot{x}^i)\tfrac{\partial L}{\partial \dot{x}^i} = -x^2 \tfrac{\partial L}{\partial \dot{x}^1} + x^1 \tfrac{\partial L}{\partial \dot{x}^2} = m(x^1\dot{x}^2 - x^2\dot{x}^1)\ , \end{aligned} \tag{3.29}$$

or

$$\mathbf{M} = \mathbf{r} \times \mathbf{P} \tag{3.30}$$

where the components of the linear momentum $\mathbf{P}$ are given by (3.25) and $\mathbf{r}$ is the particle's position from the origin.

3.3 Symmetries and Metrics

3.3.1 Metrics

From calculus we know that the distance between any two infinitesimal vectors, dx^a is defined by the line element given by

$$ds^2 = dx^a \cdot dx^b = g_{ab}\, dx^a dx^b \ . \tag{3.31}$$

In the above equation g_{ab} represents the underlying space in which dot product is taken and g^{ab} is its inverse. For our purpose we assume the metric to be degenerate, i.e. $|g_{ab}| \neq 0$. In Euclidean space in 3-dimensions, equation (3.31) takes the form

$$ds^2 = dx^2 + dy^2 + dz^2 = (dx\ dy\ dz) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{3.32}$$

where

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.33}$$

represents physical properties of the Euclidean matrix. An example of a non-Euclidean space is a sphere in which equation (3.31) yields line element given by

$$ds^2 = dr^2 + r^2\ d\theta^2 + r^2\ sin^2\theta d\phi^2 \ . \tag{3.34}$$

In the present work, we are interested in a more general metric in 4-Lorentzian space. Such metrics define distances in $n-$dimensional spaces on which lengths ds^2 can be positive, negative or zero. An example of such a space is that of Minkowski space whose metric is given by,

$$ds^2 = -dt^2 + dr^2 + r^2\, d\theta^2 + r^2\, sin^2\theta d\phi^2\ . \tag{3.35}$$

3.3.2 Killing Vectors

Geometric properties of a space-time is encoded in symmetries that are possessed by the metric 'g_{ab}' representing the space-time. These symmetries are defined in terms of Killing vectors [16]. The Killing vectors are infinitesimal generators of the symmetry group admitted by a given space-time metric. The objective of this section is not to study groups generated by Killing vectors but to understand how these vectors are determined by solving Killing equations.

A Killing vector $\underline{\xi} = \xi^a \frac{\partial}{\partial x^a}$ is the one along which Lie derivative of the metric g is zero. Mathematically, it is translated by the equation

$$\mathscr{L}_{\underline{\xi}}\, g = 0 \tag{3.36}$$

where Lie derivative of a given metric is defined by the equation

$$\mathscr{L}_{\underline{\xi}}\, g_{ab} = g_{ab,c}\, \xi^c + g_{ac}\, \xi^c_{,b} + g_{bc}\, \xi^c_{,a} \tag{3.37}$$

and "," denotes the derivative with respect to x^a $(a = 1, 2, ..., n)$. In order to show how one can find solutions of the above system of equations, we take a simple example of n-dimensional flat spacetime metric which is given by the expression,

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & -1 & 0 & . & . & . & 0 \\ . & & & & & & \\ . & & & . & & & \\ . & & & & & . & \\ 0 & 0 & 0 & . & . & . & -1 \end{pmatrix} \tag{3.38}$$

The Killing equation (3.36) in this metric takes the form

$$g_{bc}\ \xi^c_{,a} + g_{ac}\ \xi^c_{,b} = 0\ . \tag{3.39}$$

Since the metric is flat equation (3.39) can be written in a more convenient form,

$$\xi_{,ab} + \xi_{,ba} = 0, \tag{3.40}$$

that results in the set:

$$\xi_{a,b} + \xi_{b,a} = 0, \quad a \neq b \tag{3.41}$$

and

$$\xi_{,aa} = 0, \tag{3.42}$$

with no summation employed on repeated indices. Differentiating (3.41) with respect to coordinate x^a and using (3.42) gives

$$\xi_{b,aa} = 0\ . \tag{3.43}$$

Integrating the above equation twice and re-arranging indices, instantly yields

$$\xi_a = C_{ab}x^a + \alpha_a\ . \tag{3.44}$$

At this stage we require that equation (3.44) satisfies (3.40), which implies

$$C_{ab} = -C_{ba}\ . \tag{3.45}$$

It is easily noticed that there are n constants in α_n and $\frac{n(n-1)}{2}$ constants in C_{ab} giving a total of $\frac{n(n+1)}{2}$. The C_{ab} generates rotations while α_n gives n translations. This proves that the Killing vectors in a maximally symmetric space in 4-dimensions are 10.

3.4 Finding Noether Symmetries of a Metric via an Example

In the previous section we defined the Killing vectors via Lie derivative of the metric being zero. These Killing vectors give conservation laws admitted by the Lorentzian metrics. We note that the space-time metric

$$ds^2 = g_{ab}\, dx^a\, dx^b \tag{3.46}$$

can be used to construct the Lagrangian

$$L = g_{ab}\, \dot{x}^a\, \dot{x}^b\ . \tag{3.47}$$

These Lagrangians give use to Euler-Lagrange equations. From the point of view of differential equations, these Lagrangians are used to construct Noether symmetries via Noether Theorem. Our interest in the present section is a particular metric known as Bertotti-Robinson metric and how Noether symmetries of this metric are obtained.

3.4.1 Noether Symmetries of Bertotti - Robinson Metric

Bertotti - Robinson like metric (having 2-dimensional section of zero curvature) is given by the expression [2]

$$ds^2 = cosh^2(\frac{x}{a})dt^2 - dx^2 - (dy^2 + dz^2). \tag{3.48}$$

The corresponding Lagrangian is

$$L = cosh^2(\frac{x}{a})\dot{t}^2 - \dot{x}^2 - (\dot{y}^2 + \dot{z}^2). \tag{3.49}$$

The Noether symmetry generator associated with this Lagrangian is

$$X^{[1]} = \xi\frac{\partial}{\partial s} + \eta^0\frac{\partial}{\partial t} + \eta^1\frac{\partial}{\partial x} + \eta^2\frac{\partial}{\partial y} + \eta^3\frac{\partial}{\partial z} + \dot{\eta}^0\frac{\partial}{\partial \dot{t}} + \dot{\eta}^1\frac{\partial}{\partial \dot{x}} + \dot{\eta}^2\frac{\partial}{\partial \dot{y}} + \dot{\eta}^3\frac{\partial}{\partial \dot{z}}, \tag{3.50}$$

where ξ, η^0, η^1, η^2, η^3 are functions of s, t, x, y, z and $\dot{\eta}^0$, $\dot{\eta}^1$, $\dot{\eta}^2$, $\dot{\eta}^3$ are functions of s, t, x, y, z, $\dot{t}$, $\dot{x}$, $\dot{y}$, $\dot{z}$ determined by

$$\dot{\eta}^0 = D_s\eta^0 - \dot{t}D_s\xi, \tag{3.51}$$

$$\dot{\eta}^1 = D_s\eta^1 - \dot{x}D_s\xi, \tag{3.52}$$

$$\dot{\eta}^2 = D_s\eta^2 - \dot{y}D_s\xi, \tag{3.53}$$

$$\dot{\eta}^3 = D_s\eta^3 - \dot{z}D_s\xi, \tag{3.54}$$

where $D_s = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}$.

The Noether symmetry generators with a gauge function $A(s,t,x,y,z)$ can be determined using the formula

$$X^{[1]}L + LD_s\xi = D_sA.$$

The resultant over determined system of partial differential equations, after separation by monomials, is

$$\dot{t}\dot{x}^2: \qquad \xi_t = 0, \tag{3.55}$$

$$\dot{x}^3: \qquad \xi_x = 0, \tag{3.56}$$

$$\dot{x}^2\dot{y}: \qquad \xi_y = 0, \tag{3.57}$$

$$\dot{x}^2\dot{z}: \qquad \xi_z = 0, \tag{3.58}$$

$$\dot{x}^2: \qquad 2\,\eta^1_x - \xi_s = 0, \tag{3.59}$$

$$\dot{y}^2: \qquad 2\,\eta^2_y - \xi_s = 0, \tag{3.60}$$

$$\dot{z}^2: \qquad 2\,\eta^3_z - \xi_s = 0, \tag{3.61}$$

$$\dot{t}^2: \qquad \frac{2}{a}\,\eta^1 tanh(\frac{x}{a}) + 2\,\eta^0_t - \xi_s = 0, \tag{3.62}$$

$$\dot{x}\dot{y}: \qquad \eta^1_y + \eta^2_x = 0, \tag{3.63}$$

$$\dot{x}\dot{z}: \qquad \eta^1_z + \eta^3_x = 0, \tag{3.64}$$

$$\dot{y}\dot{z}: \qquad \eta^2_z + \eta^3_y = 0, \tag{3.65}$$

$$\dot{t}\dot{x}: \qquad \eta^0_x cosh^2(\frac{x}{a}) - \eta^1_t = 0, \tag{3.66}$$

$$\dot{t}\dot{y}: \qquad \eta^0_y cosh^2(\frac{x}{a}) - \eta^2_t = 0, \tag{3.67}$$

$$\dot{t}\dot{z}: \qquad \eta^0_z cosh^2(\frac{x}{a}) - \eta^3_t = 0, \tag{3.68}$$

$$\dot{t}: \qquad 2\eta^0_s cosh^2(\frac{x}{a}) - A_t = 0, \tag{3.69}$$

$$\dot{x}: \qquad 2\,\eta^1_s + A_x = 0, \tag{3.70}$$

$$\dot{y}: \qquad 2\,\eta^2_s + A_y = 0, \tag{3.71}$$

$$\dot{z}: \qquad 2\,\eta^3_s + A_z = 0, \tag{3.72}$$

$$1: \qquad A_s = 0. \tag{3.73}$$

Our target now is finding $\xi, \eta^0, \eta^1, \eta^2, \eta^3, A$ from the above system. Differentiating equations (4.24 - 4.28) with respect to ‘ s ’ gives

$$\eta^0_{ss} = \eta^1_{ss} = \eta^2_{ss} = \eta^3_{ss} = 0,$$

which immediately leads to

$$\eta^0 = \alpha_1(t,x,y,z)s + \alpha_2(t,x,y,z),$$
$$\eta^1 = \alpha_3(t,x,y,z)s + \alpha_4(t,x,y,z),$$
$$\eta^2 = \alpha_5(t,x,y,z)s + \alpha_6(t,x,y,z),$$

$$\eta^3 = \alpha_7(t,x,y,z)s + \alpha_8(t,x,y,z).$$

It is obvious from (4.10 - 4.13) that $\xi = \xi(s)$ and equation (4.14) with further analysis give

$$\eta^1 = [c_1\, x + \alpha_9(t,y,z)]s + c_2\, x + \alpha_{10}(t,y,z), \tag{3.74}$$

$$\xi = c_1 s^2 + 2c_2 s + c_3. \tag{3.75}$$

Similarly equations (4.15),(4.16) yield

$$\eta^2 = [c_1\, y + \alpha_{11}(t,x,z)]s + c_2\, y + \alpha_{12}(t,x,z), \tag{3.76}$$

$$\eta^3 = [c_1\, z + \alpha_{13}(t,x,y)]s + c_2\, z + \alpha_{14}(t,x,y). \tag{3.77}$$

Substituting (4.29), (4.31)-(4.32) in (4.18)-(4.19) leads to an expression for $\alpha_{11}, \alpha_{12}, \alpha_{13}$ and α_{14} in terms of α_9 and α_{10} :

$$\alpha_{11} = -\tfrac{\partial}{\partial y}\alpha_9\, x + \alpha_{15}(t,z)\ , \qquad \alpha_{12} = -\tfrac{\partial}{\partial y}\alpha_{10}\, x + \alpha_{16}(t,z)\ ,$$

$$\alpha_{13} = -\tfrac{\partial}{\partial z}\alpha_9\, x + \alpha_{17}(t,z)\ , \qquad \alpha_{14} = -\tfrac{\partial}{\partial z}\alpha_{10}\, x + \alpha_{18}(t,z)\ .$$

Using equation (4.20) $\alpha_9, \alpha_{10}, \alpha_{17}$, and α_{18} can be written as:

$$\alpha_9 = \alpha_{19}(t,y) + \alpha_{20}(t,z)\ , \qquad \alpha_{17} = -\tfrac{\partial}{\partial z}\alpha_{15}\, y + \alpha_{21}(t)\ ,$$

$$\alpha_{10} = \alpha_{22}(t,y) + \alpha_{23}(t,z)\ , \qquad \alpha_{18} = -\tfrac{\partial}{\partial z}\alpha_{16}\, y + \alpha_{24}(t)\ .$$

It is clear from equations (4.15) and (4.16) that $\eta^2_{xy} = \eta^3_{xz} = \eta^3_{yz} = 0$, so from equations (4.18), (4.19) and (4.20) we find, respectively, that

$$\eta^1_{yy} = \eta^1_{zz} = \eta^2_{zz} = 0,$$

which implies $\alpha_{19} = \alpha_{25}(t)\, y + \alpha_{26}(t)\ ,\quad \alpha_{22} = \alpha_{27}(t)\, y + \alpha_{28}(t)\ ,$

$$\alpha_{20} = \alpha_{29}(t)\, z + \alpha_{30}(t)\ ,\quad \alpha_{23} = \alpha_{31}(t)\, z + \alpha_{32}(t)\ ,$$

$$\alpha_{15} = \alpha_{35}(t)\, z + \alpha_{36}(t) \ , \quad \alpha_{16} = \alpha_{37}(t)\, z + \alpha_{38}(t) \ .$$

Now, we can write η^1, η^2 and η^3 with twelve unknowns of one variable ' t ' instead of six unknowns of four variables

$$\eta^1 = [c_1 x + \alpha_{25}(t) y + \alpha_{29}(t) z + \alpha_{33}(t)] s + c_2 x + \alpha_{27}(t) y + \alpha_{31}(t) z + \alpha_{34}(t), \quad (3.78)$$

$$\eta^2 = [c_1 y - \alpha_{25}(t) x + \alpha_{35}(t) z + \alpha_{36}(t)] s + c_2 y - \alpha_{27}(t) x + \alpha_{37}(t) z + \alpha_{38}(t), \quad (3.79)$$

$$\eta^3 = [c_1 z - \alpha_{29}(t) x - \alpha_{35}(t) y + \alpha_{21}(t)] s + c_2 z - \alpha_{31}(t) x - \alpha_{37}(t) y + \alpha_{24}(t). \quad (3.80)$$

At this stage we only want to express η^0 in a similar form. For this purpose we use (4.17) which shows that

$$\alpha_1 = c_1 t - \tfrac{1}{a}\, tanh(\tfrac{x}{a})[c_1 t x + y \textstyle\int \alpha_{25}(t) dt + z \int \alpha_{29}(t) dt + \int \alpha_{33}(t) dt] \ ,$$

$$\alpha_2 = c_2 t - \tfrac{1}{a}\, tanh(\tfrac{x}{a})[y \textstyle\int \alpha_{27}(t) dt + z \int \alpha_{31}(t) dt + \int \alpha_{34}(t) dt] + \beta_1(x,y,z) \ .$$

Equation (4.21) yields with some manipulation

$$\alpha_{25} = c_5\, sin(\tfrac{t}{a}) + c_6\, cos(\tfrac{t}{a}) \ , \qquad \alpha_{29} = c_8\, sin(\tfrac{t}{a}) + c_9\, cos(\tfrac{t}{a}) \ ,$$

$$\alpha_{33} = c_{11}\, sin(\tfrac{t}{a}) + c_{12}\, cos(\tfrac{t}{a}) \ , \qquad \alpha_{27} = c_{13}\, sin(\tfrac{t}{a}) + c_{14}\, cos(\tfrac{t}{a}) \ ,$$

$$\alpha_{31} = c_{15}\, sin(\tfrac{t}{a}) + c_{16}\, cos(\tfrac{t}{a}) \ , \qquad \alpha_{34} = c_{17}\, sin(\tfrac{t}{a}) + c_{18}\, cos(\tfrac{t}{a}) \ ,$$

$$c_1 = 0 \ , \qquad \beta_1 = \beta_2(y,z) \ .$$

Similarly equation (4.22) leads to

$c_5\, cos(\tfrac{t}{a}) - c_6\, sin(\tfrac{t}{a}) = 0,$ which also means $c_5\, sin(\tfrac{t}{a}) + c_6\, cos(\tfrac{t}{a}) = 0,$

$c_{13}\, cos(\tfrac{t}{a}) - c_{14}\, sin(\tfrac{t}{a}) = 0,$ which also means $c_{13}\, sin(\tfrac{t}{a}) + c_{14}\, cos(\tfrac{t}{a}) = 0,$

$$\alpha_{35}(t) = c_{19}, \qquad \alpha_{36}(t) = c_{20}, \qquad \alpha_{37}(t) = c_{22},$$

$$\alpha_{38}(t) = c_{23}\, t + c_{24} \ , \qquad \beta_1 = c_{21}\, y + \beta_4(z) \ .$$

Finally using equation (4.23) one gets

$c_8\, cos(\tfrac{t}{a}) - c_9\, sin(\tfrac{t}{a}) = 0,$ which also means $c_8\, sin(\tfrac{t}{a}) + c_9\, cos(\tfrac{t}{a}) = 0,$

$c_{15}\ cos(\frac{t}{a}) - c_{16}\ sin(\frac{t}{a}) = 0$, which also means $c_{15}\ sin(\frac{t}{a}) + c_{16}\ cos(\frac{t}{a}) = 0$,

$\alpha_{21}(t) = c_{25}$, $\qquad \alpha_{24}(t) = c_{27}$, $\qquad \beta_4 = c_{26}$.

Now, what we still need is finding the gauge function A, and to achieve this goal we come back to the first four equations that we have started with, where it turns out from equation (4.24) that:

$$A = 2a(c_{11}\ sin(\frac{t}{a}) + c_{12}\ cos(\frac{t}{a}))\ sinh(\frac{2x}{a}) + A_1(x,y,z) \tag{3.81}$$

and from equation (4.25) with some manipulation we get

$$A_1 = c_{28} + A_5(z) + A_3(y,z)\ ,$$
$$c_{11}\ cos(\tfrac{t}{a}) - c_{12}\ sin(\tfrac{t}{a}) = 0.$$

From equation (4.26) $A_3 = -2(c_{19}\ z + c_{20})\ y + A_6(z)$. For convenience let $A_5(z) + A_6(z) = A_7(z)$. Finally equation (4.27) gives

$$A_7 = -2c_{19}\ z + c_{26},$$
$$c_{19} = 0.$$

Inserting all these results in equation (4.36) we find,

$$A = c_{29} - 2c_{20}\ y - 2c_{25}\ z. \tag{3.82}$$

In the light of the above results we can write ξ, η^0, η^1, η^2, $\eta^3$and A explicitly as:

$$\xi = 2c_2\ s + c_3, \tag{3.83}$$
$$\eta^0 = c_2 t + (c_{17}\ cos(\frac{t}{a}) - c_{18}\ sin(\frac{t}{a}))\ tanh(\frac{x}{a}) + c_{21}\ y + c_{26}, \tag{3.84}$$
$$\eta^1 = c_2 x + c_{17}\ sin(\frac{t}{a}) + c_{18}\ cos(\frac{t}{a}), \tag{3.85}$$

$$\eta^2 = c_{20}\ s + c_2\ y + c_{22}\ z + c_{23}\ t + c_{24}, \tag{3.86}$$

$$\eta^3 = c_{25}\ s + c_2\ z - c_{22}\ y + c_{27}, \tag{3.87}$$

$$A = c_{29} - 2c_{20}\ y - 2c_{25}\ z. \tag{3.88}$$

In order to check conversely we satisfy equations (4.10 - 4.28). This implies that $c_2 = 0$, $c_{21} = c_{23} = 0$. Renumbering the constants c_i's, we finally have the solution of the system (4.10 - 4.28) as :

$$\xi = C_1, \tag{3.89}$$

$$\eta^0 = (C_2\ cos(\frac{t}{a}) - C_3\ sin(\frac{t}{a}))\ tanh(\frac{x}{a}) + C_4, \tag{3.90}$$

$$\eta^1 = C_2\ sin(\frac{t}{a}) + C_3\ cos(\frac{t}{a}), \tag{3.91}$$

$$\eta^2 = C_5\ s + C_6\ z + C_7, \tag{3.92}$$

$$\eta^3 = C_8\ s - C_6\ y + c_9, \tag{3.93}$$

$$A = C_{10} - 2C_5\ y - 2C_8\ z\ . \tag{3.94}$$

Thus, we have the following 9 non-trivial Noether symmetries :

$\mathbf{X_1} = \frac{\partial}{\partial s}$, with gauge term $f = 0$,

$\mathbf{X_2} = cos(\frac{t}{a})\ tanh(\frac{x}{a})\ \frac{\partial}{\partial t} + sin(\frac{t}{a})\ \frac{\partial}{\partial x}$, with gauge term $f = 0$,

$\mathbf{X_3} = -sin(\frac{t}{a})\ tanh(\frac{x}{a})\ \frac{\partial}{\partial t} + cos(\frac{t}{a})\ \frac{\partial}{\partial x}$, with gauge term $f = 0$,

$\mathbf{X_4} = \frac{\partial}{\partial t}$, with gauge term $f = 0$,

$\mathbf{X_5} = s\frac{\partial}{\partial y}$, with gauge term $f = -2y$,

$\mathbf{X_6} = z\ \frac{\partial}{\partial y} - y\ \frac{\partial}{\partial z}$, with gauge term $f = 0$,

$\mathbf{X_7} = \frac{\partial}{\partial y}$, with gauge term $f = 0$,

$\mathbf{X_8} = s\,\frac{\partial}{\partial z}$, with gauge term $f = -2z$,

$\mathbf{X_9} = \frac{\partial}{\partial z}$, with gauge term $f = 0$.

Chapter 4

Noether and Lie Symmetries Associated with Milne Metric

The connection between symmetry and conservation laws is inherent in mathematical physics since Emmy Noether published, in 1918, her hugely influential work linking the two. Noether proved that for every symmetry generator, which is admitted by invariance of the action integral for a Lagrangian system, there exist a conservation law.

In this chapter, we investigate Noether symmetries of the geodesic equations of a Lorentzian metric, known as Milne model which represents an empty universe and is of interest in general Relativity for being a special case of a well known Friedmann Lemaî̂tre Robertson Walker metric [16] . We also extend our investigations by writing the wave equation in Milne spacetime and then find its Lie point symmetries.

The number of Lie symmetries of the wave equation is compared with the Milne spacetime's isometries and classical wave equation in 3-dimensions.

4.1 Noether Symmetries and Conservation Laws of Milne Metric

Geodesic equations are the Euler Lagrange equations determined from invariance of an action integral [3]. In order to find Noether symmetries admitted by the geodesic equations for the Milne metric

$$ds^2 = -dt^2 + t^2(dx^2 + e^{2x}(dy^2 + dz^2)) \tag{4.1}$$

we write a Lagrangian $L = L(s,t,x,y,z,\dot{t},\dot{x},\dot{y},\dot{z})$ that can be constructed by the Milne metric and given by the expression

$$L = -\dot{t}^2 + t^2(\dot{x}^2 + e^{2x}(\dot{y}^2 + \dot{z}^2)). \tag{4.2}$$

The general Noether symmetry generator corresponding to this Lagrangian is

$$X = \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial t} + \eta^2\frac{\partial}{\partial x} + \eta^3\frac{\partial}{\partial y} + \eta^4\frac{\partial}{\partial z} + \dot{\eta}^1\frac{\partial}{\partial \dot{t}} + \dot{\eta}^2\frac{\partial}{\partial \dot{x}} + \dot{\eta}^3\frac{\partial}{\partial \dot{y}} + \dot{\eta}^4\frac{\partial}{\partial \dot{z}} \tag{4.3}$$

where the infinitesimals ξ, η^1,η^2,η^3,η^4 are functions of s,t,x,y,z, given by

$$\dot{\eta}^1 = D_s\eta^1 - \dot{t}D_s\xi, \tag{4.4}$$

$$\dot{\eta}^2 = D_s\eta^2 - \dot{x}D_s\xi, \tag{4.5}$$

$$\dot{\eta}^3 = D_s\eta^3 - \dot{y}D_s\xi, \tag{4.6}$$

$$\dot{\eta}^4 = D_s\eta^4 - \dot{z}D_s\xi, \tag{4.7}$$

and $D_s = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}$.

Using the Noetherian criterion

$$XL + LD_s\xi = D_s f, \tag{4.8}$$

with a gauge function $f(s,t,x,y,z)$, one can write

$$\begin{aligned}&[\xi\tfrac{\partial}{\partial s} + \eta^1\tfrac{\partial}{\partial t} + \eta^2\tfrac{\partial}{\partial x} + \eta^3\tfrac{\partial}{\partial y} + \eta^4\tfrac{\partial}{\partial z} + \dot{\eta}^1\tfrac{\partial}{\partial \dot{t}} + \dot{\eta}^2\tfrac{\partial}{\partial \dot{x}} + \dot{\eta}^3\tfrac{\partial}{\partial \dot{y}}\\&+\dot{\eta}^4\tfrac{\partial}{\partial \dot{z}}][-\dot{t}^2 + t^2(\dot{x}^2 + e^{2x}(\dot{y}^2 + \dot{z}^2))] + [-\dot{t}^2 + t^2(\dot{x}^2 + e^{2x}\\&(\dot{y}^2 + \dot{z}^2))][\tfrac{\partial}{\partial s} + \dot{t}\tfrac{\partial}{\partial t} + \dot{x}\tfrac{\partial}{\partial x} + \dot{y}\tfrac{\partial}{\partial y} + \dot{z}\tfrac{\partial}{\partial z}]\xi = [\tfrac{\partial}{\partial s} + \dot{t}\tfrac{\partial}{\partial t} + \dot{x}\tfrac{\partial}{\partial x}\\&+\dot{y}\tfrac{\partial}{\partial y} + \dot{z}\tfrac{\partial}{\partial z}]f\end{aligned} \tag{4.9}$$

Simplifying (4.9) and comparing the coefficients of different powers of $\dot{t},\dot{x},\dot{y},\dot{z}$ yield the following system of 19 differential equations :

$$\dot{t}^3: \qquad \xi_t = 0, \tag{4.10}$$

$$\dot{t}^2\dot{x}: \qquad \xi_x = 0, \tag{4.11}$$

$$\dot{t}^2\dot{y}: \qquad \xi_y = 0, \tag{4.12}$$

$$\dot{t}^2\dot{z}: \qquad \xi_z = 0, \tag{4.13}$$

$$\dot{t}^2: \qquad 2\ \eta^1_t - \ \xi_s = 0, \tag{4.14}$$

$$\dot{x}^2: \qquad 2\ \eta^1 + 2\ t\ \eta^2_x - t\ \xi_s = 0, \tag{4.15}$$

$$\dot{y}^2: \qquad 2\ \eta^1 + 2\ t\ \eta^2 + 2\ t\ \eta^3_y - t\ \xi_s = 0, \tag{4.16}$$

$$\dot{z}^2: \qquad 2\ \eta^1 + 2\ t\ \eta^2 + 2\ t\ \eta^4_z - t\ \xi_s = 0, \tag{4.17}$$

$$\dot{t}\dot{x}: \qquad \eta^1_x - t^2\ \eta^2_t = 0, \tag{4.18}$$

$$\dot{t}\dot{y}: \qquad \eta^1_y - t^2 e^{2x}\ \eta^3_t = 0, \tag{4.19}$$

$$\dot{t}\dot{z}: \qquad \eta^1_z - t^2 e^{2x}\ \eta^4_t = 0, \tag{4.20}$$

$$\dot{x}\dot{y}: \qquad \eta^2_y + e^{2x}\ \eta^3_x = 0, \tag{4.21}$$

$$\dot{x}\dot{z}: \qquad \eta^2_z + e^{2x}\ \eta^4_x = 0, \tag{4.22}$$

$$\dot{y}\dot{z}: \qquad \eta^3_z + \ \eta^4_y = 0, \tag{4.23}$$

$$\dot{t}: \qquad 2\ \eta^1_s + f_t = 0, \tag{4.24}$$

$$\dot{x}: \qquad 2\ t^2\ \eta^2_s - f_x = 0, \tag{4.25}$$

$$\dot{y}: \qquad 2\ t^2 e^{2x}\ \eta^3_s - f_y = 0, \tag{4.26}$$

$$\dot{z}: \qquad 2\ t^2 e^{2x}\ \eta^4_s - f_z = 0, \tag{4.27}$$

$$1: \qquad f_s = 0. \tag{4.28}$$

From equations (4.10 - 4.13) it can be easily seen that ξ is a function of s only, i.e. $\xi = \xi(s)$. Similarly from equation (4.28), we find that $f = f(t,x,y,z)$. Moreover, from Equations (4.24 - 4.28) we find that η^1,η^2,η^3,η^4 take the form:

$$\eta^1 = a_1(t,x,y,z)s + a_2(t,x,y,z), \tag{4.29}$$

$$\eta^2 = a_3(t,x,y,z)s + a_4(t,x,y,z), \tag{4.30}$$

$$\eta^3 = a_5(t,x,y,z)s + a_6(t,x,y,z), \tag{4.31}$$

$$\eta^4 = a_7(t,x,y,z)s + a_8(t,x,y,z). \tag{4.32}$$

Consequently, by equation (4.14) one obtains,

$$\xi = \frac{\partial a_1}{\partial t}s^2 + 2\frac{\partial a_2}{\partial t}s + C_3. \tag{4.33}$$

Keeping in mind that ξ, a_1 and a_2 are independent of s, we get $\frac{\partial^3 \xi}{\partial s^3} = 0$ which means that:

$$\xi = \frac{1}{2}C_1 s^2 + C_2 s + C_3. \tag{4.34}$$

Comparing equation (4.33) with (4.34) yields

$$a_1 = \tfrac{1}{2}C_1\, t + a_9(x,y,z),$$
$$a_2 = \tfrac{1}{2}C_2\, t + a_{10}(x,y,z).$$

Substituting these values of a_1 and a_2 in equation (4.29) gives

$$\eta^1 = (\frac{1}{2}C_1\, t + a_9(x,y,z))s + \frac{1}{2}C_2\, t + a_{10}(x,y,z). \tag{4.35}$$

Now, from equation (4.18) we get

$$\tfrac{\partial}{\partial t}(a_3)s+\tfrac{\partial}{\partial t}a_4=\tfrac{1}{t^2}(\tfrac{\partial}{\partial x}(a_9)s+\tfrac{\partial}{\partial x}a_{10}).$$

Integrating the above equation with respect to 't' and comparing the coefficients of 's' lead to:

$$a_3=-\tfrac{1}{t}\tfrac{\partial}{\partial x}a_9,$$
$$a_4=-\tfrac{1}{t}\tfrac{\partial}{\partial x}a_{10}+a_{11}(x,y,z).$$

In the light of the above results, η^2 takes the form:

$$\eta^2=-\frac{1}{t}\frac{\partial a_9}{\partial x}s-\frac{1}{t}\frac{\partial a_{10}}{\partial x}+a_{11}(x,y,z). \tag{4.36}$$

Similarly, equations (4.19),(4.20), respectively, give

$$\eta^3=-\frac{1}{t}e^{-2x}\frac{\partial a_9}{\partial y}s-\frac{1}{t}e^{-2x}\frac{\partial a_{10}}{\partial y}+a_{12}(x,y,z)\ , \tag{4.37}$$
$$\eta^4=-\frac{1}{t}e^{-2x}\frac{\partial a_9}{\partial z}s-\frac{1}{t}e^{-2x}\frac{\partial a_{10}}{\partial z}+a_{13}(x,y,z). \tag{4.38}$$

Using these expressions of η^3 and η^4 in equation (4.23), we obtain the following partial differential equation:

$$-\frac{2}{t}e^{-2x}\frac{\partial^2 a_9}{\partial y\partial z}s-\frac{2}{t}e^{-2x}\frac{\partial^2 a_{10}}{\partial y\partial z}+\frac{\partial a_{12}}{\partial z}+\frac{\partial a_{13}}{\partial y}=0. \tag{4.39}$$

Differentiating with respect to 's' leads to,

$$\frac{\partial^2 a_9}{\partial y\partial z}=0, \tag{4.40}$$

which has the following solution:

$$a_9 = a_{14}(x,y) + a_{15}(x,z). \tag{4.41}$$

Inserting equation (4.40) in (4.39) and differentiation with respect to 't' yield

$$a_{10} = a_{16}(x,y) + a_{17}(x,z). \tag{4.42}$$

By substituting equations (4.41),(4.42) in (4.39) and integrating it over 'y', we have

$$a_{13} = -\int \tfrac{\partial a_{12}}{\partial z} dy + a_{18}(x,z).$$

Now, using the above values of a_9, a_{10} and a_{13} in equations (4.35 - 4.38), η^1, η^2, η^3 and η^4 take the form:

$$\eta^1 = (\frac{1}{2}C_1\, t + a_{14}(x,y) + a_{15}(x,z))s + \frac{1}{2}C_2\, t + a_{16}(x,y) + a_{17}(x,z)\ , \tag{4.43}$$

$$\eta^2 = -\frac{1}{t}(\frac{\partial a_{14}}{\partial x} + \frac{\partial a_{15}}{\partial x})s - \frac{1}{t}(\frac{\partial a_{16}}{\partial x} + \frac{\partial a_{17}}{\partial x}) + a_{11}(x,y,z)\ , \tag{4.44}$$

$$\eta^3 = -\frac{1}{t}e^{-2x}\frac{\partial a_{14}}{\partial y}s - \frac{1}{t}e^{-2x}\frac{\partial a_{16}}{\partial y} + a_{12}(x,y,z)\ , \tag{4.45}$$

$$\eta^4 = -\frac{1}{t}e^{-2x}\frac{\partial a_{15}}{\partial z}s - \frac{1}{t}e^{-2x}\frac{\partial a_{17}}{\partial z} - \int \frac{\partial a_{12}}{\partial z}dy + a_{18}(x,z)\ . \tag{4.46}$$

Next, finding the partial derivative of η^2 with respect to 'y' and the partial derivative of η^3 with respect to 'x' and inserting them in equation (4.21) lead to

$$2(\frac{\partial^2 a_{14}}{\partial x \partial y} - \frac{\partial a_{14}}{\partial y})s + 2\frac{\partial^2 a_{16}}{\partial x \partial y} - 2\frac{\partial a_{16}}{\partial y} - t\frac{\partial a_{11}}{\partial y} - te^{2x}\frac{\partial a_{12}}{\partial x} = 0, \tag{4.47}$$

which contains four different functions need to be found explicitly or in terms of each other. To achieve this task, we differentiate this equation with respect to 's' and solve the resulting PDE to have

$$a_{14} = a_{19}(x) + a_{20}(y)e^x.$$

Also by differentiating equation (4.47) with respect to 't' and solving the resulting PDE, one has

$$a_{11} = -e^{2x}\int \frac{\partial a_{12}}{\partial x} dy + a_{21}(x,z).$$

Consequently, we get from equation (4.47) that

$$a_{16} = a_{22}(x) + a_{23}(y)e^x.$$

Similarly, substituting η^2 and η^4 in equation (4.22) yields

$$\begin{aligned} &2(\tfrac{\partial^2 a_{15}}{\partial x \partial z} - \tfrac{\partial a_{15}}{\partial z})s + 2\tfrac{\partial^2 a_{17}}{\partial x \partial z} - t\tfrac{\partial a_{17}}{\partial z} - t\tfrac{\partial a_{21}}{\partial z} - te^{2x}\tfrac{\partial a_{18}}{\partial x} \\ &+ te^{2x}(\tfrac{\partial}{\partial z}\int \tfrac{\partial a_{12}}{\partial x} dy + \tfrac{\partial}{\partial x}\int \tfrac{\partial a_{12}}{\partial z} dy) = 0. \end{aligned} \tag{4.48}$$

As we have done with equation (4.47), taking the partial derivatives of (4.48) with respect to 's','y','t' and solving the resulting PDEs give

$$a_{15} = a_{24}(x) + a_{25}(z)e^x,$$

$$a_{12} = a_{26}(x,y) + a_{27}(y,z),$$
$$a_{21} = -e^{2x} \int \tfrac{\partial a_{18}}{\partial x} dz + a_{28}(x),$$
$$a_{17} = a_{29}(x) + a_{30}(z)e^{x}.$$

Now, in the light of the above values of the functions a_{11}, a_{12}, $(a_{14} - a_{17})$ and a_{21}, we can rewrite equation (4.43 - 4.46) as:

$$\eta^1 = (\tfrac{1}{2}C_1\, t + a_{31}(x) + (a_{20}(y) + a_{25}(z))e^{x})s + \tfrac{1}{2}C_2\, t + a_{32}(x) + (a_{23}(y) + a_{30}(z))e^{x}, \tag{4.49}$$

$$\eta^2 = -\tfrac{1}{t}(\tfrac{\partial a_{31}}{\partial x} + (a_{20} + a_{25})e^{x})s - \tfrac{1}{t}(\tfrac{\partial a_{32}}{\partial x} + (a_{23} + a_{30})e^{x}) - e^{2x} \int \tfrac{\partial a_{26}}{\partial x} dy - e^{2x} \int \tfrac{\partial a_{18}}{\partial x} dz + a_{28}(x), \tag{4.50}$$

$$\eta^3 = -\frac{1}{t}e^{-x}\frac{\partial a_{20}}{\partial y}s - \frac{1}{t}e^{-x}\frac{\partial a_{23}}{\partial y} + a_{26}(x,y) + a_{27}(y,z), \tag{4.51}$$

$$\eta^4 = -\frac{1}{t}e^{-x}\frac{\partial a_{25}}{\partial z}s - \frac{1}{t}e^{-x}\frac{\partial a_{30}}{\partial z} - \int \frac{\partial a_{27}}{\partial z} dy + a_{18}(x,z), \tag{4.52}$$

where we write for convenience $a_{19}(x) + a_{24}(x)$ as $a_{31}(x)$ and $a_{22}(x) + a_{29}(x)$ as $a_{32}(x)$. Next, Substituting equations (4.34), (4.49) and (4.50) in equation (4.15) leads to

$$(a_{31} - \tfrac{\partial^2 a_{31}}{\partial x^2})s + a_{32} - \tfrac{\partial^2 a_{32}}{\partial x^2} - te^{2x}\tfrac{\partial}{\partial x}\int \tfrac{\partial a_{26}}{\partial x} dy - 2te^{2x}\int \tfrac{\partial a_{26}}{\partial x} dy - te^{2x}\tfrac{\partial}{\partial x}\int \tfrac{\partial a_{18}}{\partial x} dz - 2te^{2x}\int \tfrac{\partial a_{18}}{\partial x} dz + 2t\tfrac{\partial a_{28}}{\partial x} = 0. \tag{4.53}$$

Differentiating (4.53) with respect to s, y, z, t and solving the resulting PDEs respectively yield the following:

$$a_{31} = c_4 e^x + c_5 e^{-x},$$

$$a_{26} = a_{33}(y) + a_{34}(y)e^{-2x},$$

$$a_{18} = a_{35}(z) + a_{36}(z)e^{-2x},$$

$$a_{28} = c_6,$$

$$a_{32} = c_7 e^x + c_8 e^{-x}.$$

Using these results in equations (4.49) - (4.51) and substituting η^1,η^2 and η^3 in equation (4.16) lead to:

$$\begin{aligned} &e^{-x}(2c_5 - \tfrac{\partial^2 a_{20}}{\partial y^2})s + 2c_8 e^{-x} + c_6 t - e^{-x}\tfrac{\partial^2 a_{23}}{\partial y^2} + 2t\int a_{34}dy + te^{-2x}\tfrac{\partial a_{34}}{\partial y} \\ &+ 2t\int a_{36}dz + t\tfrac{\partial a_{33}}{\partial y} + t\tfrac{\partial a_{27}}{\partial y} = 0. \end{aligned} \tag{4.54}$$

As we have done before with equation (4.53) differentiation and simplification yield

$$a_{20} = c_5 y^2 + c_9 y + c_{10},$$

$$a_{23} = c_8 y^2 + c_{12} y + c_{13},$$

$$a_{27} = a_{37}(z)y + a_{38}(z) + a_{39}(y),$$

$$a_{33} = -a_{39} - c_{11} y^2 - c_6 y + c_{14},$$

$$a_{34} = c_{11},$$

$$a_{36} = -\tfrac{1}{2}\tfrac{\partial}{\partial z} a_{37}.$$

In the light of the above, equation (4.17) takes the form

$$\begin{aligned} &2e^{-x}(2c_5 - \tfrac{\partial^2 a_{25}}{\partial z^2})s + 4c_8 e^{-x} + 4c_{11} ty - 2ta_{37} + 2tc_6 - 2e^{-x}\tfrac{\partial^2 a_{30}}{\partial z^2} \\ &-t\tfrac{\partial^2 a_{37}}{\partial z^2} y^2 - 2t\tfrac{\partial^2 a_{38}}{\partial z^2} y + 2t\tfrac{\partial a_{35}}{\partial z} - t\tfrac{\partial^2 a_{37}}{\partial z^2} = 0 \end{aligned} \tag{4.55}$$

which implies that

$$a_{25} = c_5 z^2 + c_{16} z + c_{17},$$

$$a_{30} = c_8 z^2 + c_{18} z + c_{19},$$

$$a_{35} = \frac{1}{2} c_{20} z^2 - c_{27} z + c_{24},$$

$$c_{27} = c_6 - c_{21},$$

$$a_{37} = c_{20} z + c_{21},$$

$$a_{38} = c_{11} z^2 + c_{22} z + c_{23}.$$

Finally, we obtain the following expressions for ξ, η^1, η^2, η^3 and η^4 :

$$\xi = \frac{1}{2}\, C_1\, s^2 + C_2\, s + C_3, \tag{4.56}$$

$$\begin{aligned} \eta^1 = {} & \tfrac{1}{2} ts\, C_1 + \tfrac{1}{2} t\, C_2 - \tfrac{1}{2} s(e^x(y^2+z^2) + e^{-x})\, C_5 - \tfrac{1}{2} sze^x\, C_6 \\ & - \tfrac{1}{2} se^x\, C_7 - \tfrac{1}{2} sye^x\, C_8 + (e^{-x} + (y^2+z^2)e^x)\, C_9 + ze^x\, C_{10} \\ & + e^x\, C_{11} + ye^x\, C_{12}, \end{aligned} \tag{4.57}$$

$$\begin{aligned} \eta^2 = {} & \tfrac{1}{2t}((y^2+z^2)se^x - se^{-x})\, C_5 + \tfrac{1}{2t} sze^x\, C_6 + \tfrac{1}{2t} se^x\, C_7 + \tfrac{1}{2t} sye^x\, C_8 \\ & + \tfrac{1}{t}(e^{-x} - (y^2+z^2)e^x)\, C_9 - \tfrac{1}{t} ze^x\, C_{10} - \tfrac{1}{t} e^x\, C_{11} - \tfrac{1}{t} ye^x\, C_{12} \\ & + 2y\, C_{13} - z\, C_{14} - C_{15}, \end{aligned} \tag{4.58}$$

$$\begin{aligned} \eta^3 = {} & \tfrac{1}{t} sye^{-x}\, C_5 + \tfrac{1}{2t} se^{-x}\, C_8 - \tfrac{2}{t} ye^{-x}\, C_9 - \tfrac{1}{t} e^{-x}\, C_{12} + (e^{-2x} - y^2 \\ & + z^2)\, C_{13} + yz\, C_{14} + y\, C_{15} + z\, C_{16} + C_{17}, \end{aligned} \tag{4.59}$$

$$\begin{aligned} \eta^4 = {} & \tfrac{1}{t} sze^{-x}\, C_5 + \tfrac{1}{2}\tfrac{1}{t} se^{-x}\, C_6 - \tfrac{2}{t} ze^{-x}\, C_9 - \tfrac{1}{t} e^{-x}\, C_{10} - 2yz\, C_{13} \\ & - \tfrac{1}{2}(e^{-2x} + y^2 - z^2)\, C_{14} + z\, C_{15} - y\, C_{16} + C_{18}, \end{aligned} \tag{4.60}$$

where C_i's are arbitrary constants. At this stage, we only want to find an explicit form for the gauge term f. This can be done easily by substituting equations (4.57) - (4.60) in (4.24) - (4.28):

$$f=-\frac{1}{2}t^2\,C_1+C_4+t(e^{-x}+(y^2+z^2)e^x)\,C_5+tze^x\,C_6+te^x\,C_7+tye^x\,C_8. \qquad (4.61)$$

The solution of the overdetermined linear system determines the following entire set of 17 non-trivial Noether symmetries:

$\mathbf{X_1}=\frac{1}{2}s^2\,\frac{\partial}{\partial s}+\frac{1}{2}st\,\frac{\partial}{\partial t}$, with gauge term $f=-\frac{1}{2}t^2$

$\mathbf{X_2}=s\,\frac{\partial}{\partial s}+\frac{1}{2}t\,\frac{\partial}{\partial t}$,

$\mathbf{X_3}=\frac{\partial}{\partial s}$,

$\mathbf{X_4}=-\frac{1}{2}s(e^x(y^2+z^2)+e^{-x})\,\frac{\partial}{\partial t}+\frac{1}{2t}((y^2+z^2)se^x-se^{-x})\,\frac{\partial}{\partial x}+\frac{1}{t}sye^{-x}\,\frac{\partial}{\partial y}+\frac{1}{t}sze^{-x}\,\frac{\partial}{\partial z}$,

with gauge term $f=t(e^{-x}+(y^2+z^2)e^x)$,

$\mathbf{X_5}=-\frac{1}{2}sze^x\,\frac{\partial}{\partial t}+\frac{1}{2t}sze^x\,\frac{\partial}{\partial x}+\frac{1}{2t}se^{-x}\,\frac{\partial}{\partial z}$, with gauge term $f=tze^x$,

$\mathbf{X_6}=-\frac{1}{2}se^x\,\frac{\partial}{\partial t}+\frac{1}{2t}se^x\,\frac{\partial}{\partial x}$, with gauge term $f=te^x$,

$\mathbf{X_7}=-\frac{1}{2}sye^x\,\frac{\partial}{\partial t}+\frac{1}{2t}sye^x\,\frac{\partial}{\partial x}+\frac{1}{2t}se^{-x}\,\frac{\partial}{\partial y}$, with gauge term $f=tye^x$,

$\mathbf{X_8}=(e^{-x}+(y^2+z^2)e^x)\,\frac{\partial}{\partial t}+\frac{1}{t}(e^{-x}-(y^2+z^2)e^x)\,\frac{\partial}{\partial x}-\frac{2}{t}ye^{-x}\,\frac{\partial}{\partial y}-\frac{2}{t}ze^{-x}\,\frac{\partial}{\partial z}$,

$\mathbf{X_9}=ze^x\,\frac{\partial}{\partial t}-\frac{1}{t}ze^x\,\frac{\partial}{\partial x}-\frac{1}{t}e^{-x}\,\frac{\partial}{\partial z}$,

$\mathbf{X_{10}}=e^x\,\frac{\partial}{\partial t}-\frac{1}{t}e^x\,\frac{\partial}{\partial x}$,

$\mathbf{X_{11}}=ye^x\,\frac{\partial}{\partial t}-\frac{1}{t}ye^x\,\frac{\partial}{\partial x}-\frac{1}{t}e^{-x}\,\frac{\partial}{\partial y}$,

$\mathbf{X_{12}}=2y\,\frac{\partial}{\partial x}+(e^{-2x}-y^2+z^2)\,\frac{\partial}{\partial y}-2yz\,\frac{\partial}{\partial z}$,

$\mathbf{X_{13}}=-z\,\frac{\partial}{\partial x}+yz\,\frac{\partial}{\partial y}-\frac{1}{2}(e^{-2x}+y^2-z^2)\,\frac{\partial}{\partial z}$,

$\mathbf{X_{14}}=-\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}+z\,\frac{\partial}{\partial z}$,

$\mathbf{X_{15}}=z\,\frac{\partial}{\partial y}-y\,\frac{\partial}{\partial z}$, $\quad\mathbf{X_{16}}=\frac{\partial}{\partial y}$, $\quad\mathbf{X_{17}}=\frac{\partial}{\partial z}$.

The commutation relations of the Lie algebra of the 17 Noether symmetries are shown in Table (4.1):

[,]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}
X_1	0	$-X_1$	$-X_2$	0	0	0	0	X_4	X_5	X_6	X_7	0	0	0	0	0	0
X_2	X_1	0	$-X_3$	$\frac{1}{2}X_4$	$\frac{1}{2}X_5$	$\frac{1}{2}X_6$	$\frac{1}{2}X_7$	$-\frac{1}{2}X_8$	$-\frac{1}{2}X_9$	$-\frac{1}{2}X_{10}$	$-\frac{1}{2}X_{11}$	0	0	0	0	0	0
X_3	X_2	X_3	0	$-\frac{1}{2}X_8$	$-\frac{1}{2}X_9$	$-\frac{1}{2}X_{10}$	$-\frac{1}{2}X_{11}$	0	0	0	0	0	0	0	0	0	0
X_4	0	$-\frac{1}{2}X_4$	$\frac{1}{2}X_8$	0	0	0	0	0	0	0	0	0	0	$-X_4$	0	$-2X_7$	$-2X_5$
X_5	0	$-\frac{1}{2}X_5$	$\frac{1}{2}X_9$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}X_4$	0	X_7	0	$-X_6$
X_6	0	$-\frac{1}{2}X_6$	$\frac{1}{2}X_{10}$	0	0	0	0	0	0	0	0	$-2X_7$	X_5	X_6	0	0	0
X_7	0	$-\frac{1}{2}X_7$	$\frac{1}{2}X_{11}$	0	0	0	0	0	0	0	0	$-X_4$	0	0	$-X_5$	$-X_6$	0
X_8	$-X_4$	$\frac{1}{2}X_8$	0	0	0	0	0	0	0	0	0	0	0	$-X_8$	0	$-2X_{11}$	$-2X_9$
X_9	$-X_5$	$\frac{1}{2}X_9$	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}X_8$	0	X_{11}	0	$-X_{10}$
X_{10}	$-X_6$	$\frac{1}{2}X_{10}$	0	0	0	0	0	0	0	0	0	$-2X_{11}$	X_9	X_{10}	0	0	0
X_{11}	$-X_7$	$\frac{1}{2}X_{11}$	0	0	0	0	0	0	0	0	0	$-X_8$	0	0	$-X_9$	$-X_{10}$	0
X_{12}	0	0	0	0	0	$2X_7$	X_4	0	0	$2X_{11}$	X_8	0	0	$-X_{12}$	$2X_{13}$	$2X_{14}$	$-2X_{15}$
X_{13}	0	0	0	0	$-\frac{1}{2}X_4$	$-X_5$	0	0	$-\frac{1}{2}X_8$	$-X_9$	0	0	0	$-X_{13}$	$-\frac{1}{2}X_{12}$	$-X_{15}$	$-X_{14}$
X_{14}	0	0	0	X_4	0	$-X_6$	0	X_8	0	$-X_{10}$	0	X_{12}	X_{13}	0	0	$-X_{16}$	$-X_{17}$
X_{15}	0	0	0	0	$-X_7$	0	X_5	0	$-X_{11}$	0	X_9	$-2X_{13}$	$\frac{1}{2}X_{12}$	0	0	X_{17}	$-X_{16}$
X_{16}	0	0	0	$2X_7$	0	0	X_6	$2X_{11}$	0	0	X_{10}	$-2X_{14}$	X_{15}	X_{16}	$-X_{17}$	0	0
X_{17}	0	0	0	$2X_5$	X_6	0	0	$2X_9$	X_{10}	0	0	$2X_{15}$	X_{14}	X_{17}	X_{16}	0	0

Table 4.1: Algebra of Noether symmetries associated with Milne metric

Each of these yields a conservation law (first integral) of the geodesic equations via Noether's theorem; 7 more than those given by the Killing vectors/isometries [16] of the Milne metric (the additional seven generators are $\mathbf{X_1}$ to $\mathbf{X_7}$). For example, a conserved quantity, associated with $\mathbf{X_3}$ is

$$\begin{aligned} T &= (\xi\dot{t}-\eta^1)\frac{\partial L}{\partial \dot{t}} + (\xi\dot{x}-\eta^2)\frac{\partial L}{\partial \dot{x}} + (\xi\dot{y}-\eta^3)\frac{\partial L}{\partial \dot{y}} + (\xi\dot{z}-\eta^4)\frac{\partial L}{\partial \dot{z}} - \xi L + f(s,t,x,y,z) \\ &= sL + t\dot{t} \end{aligned} \tag{4.62}$$

4.2 Wave Equation on Milne Metric and its Lie Symmetries

In order to discuss the wave equation on the Milne metric, we write the wave equation using the formula

$$\Box\, u = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}\; g^{ij}\frac{\partial u}{\partial x^i}) = 0$$

where g^{ij} is the inverse and g is the determinant of the metric. Simplifying this expression the wave equation takes the simple form,

$$u_{tt} - \frac{1}{t^2}u_{xx} - \frac{1}{t^2e^{2x}}u_{yy} - \frac{1}{t^2e^{2x}}u_{zz} = -\frac{3}{t}u_t + \frac{2}{t^2}u_x \qquad (4.63)$$

To find the Lie point symmetries of the wave equation we use a criterion that determines Lie point symmetries given by

$$\mathscr{X}\,((4.63)|_{(4.63)}) = 0 \qquad (4.64)$$

where X is the prolonged symmetry generator given by

$$\mathscr{X} = \xi^1\partial_t + \xi^2\partial_x + \xi^3\partial_y + \xi^4\partial_z + \eta\,\partial_u. \qquad (4.65)$$

Solving equation(4.64) gives a set of determining equations to be solved for the four unknowns determining Lie point symmetries:

$$
\begin{aligned}
&\xi^1{}_u = 0,\ \ \xi^1{}_y - t^2e^{2x}\xi^3{}_t = 0,\ \ \xi^3{}_u = 0,\\
&\xi^1{}_z - t^2e^{2x}\xi^4{}_t = 0,\ \ \xi^3{}_y - \xi^4{}_z = 0,\\
&t^2\xi^1{}_{tt} - \xi^1{}_x - t^2\xi^4{}_{tz} + \xi^1 - t\xi^1{}_t = 0,\\
&t\xi^1{}_{tx} - \xi^1{}_x - t\xi^4{}_{xz} + \xi^1 - t\xi^1{}_t = 0,\\
&\xi^1{}_{xx} - t^2\xi^4{}_{tz} - \xi^1{}_x = 0,\ \ \xi^3{}_z + \xi^4{}_y = 0,\\
&t^2e^{2x}\xi^3{}_{tt} + e^{2x}\xi^3{}_x + te^{2x}\xi^3{}_t - \xi^4{}_{yz} = 0,\\
&\xi^3{}_{tx} + \xi^3{}_t = 0,\ \ \xi^4{}_{tx} + \xi^4{}_t = 0,\\
&e^{2x}\xi^3{}_{xx} + \xi^4{}_{yz} + e^{2x}\xi^3{}_x = 0,\ \ \xi^4{}_u = 0,\\
&t^2e^{2x}\xi^4{}_{tt} + e^{2x}\xi^4{}_x + te^{2x}\xi^4{}_t - \xi^4{}_{zz} = 0,\\
&e^{2x}\xi^4{}_{xx} + \xi^4{}_{zz} + e^{2x}\xi^4{}_x = 0,\ \ \xi^4{}_{ty} = 0,\\
&\xi^4{}_{xy} = 0,\ \ \xi^4{}_{yy} + \xi^4{}_{zz} = 0,\ \ \xi^4{}_{tzz} = 0,\\
&\xi^4{}_{xzz} = 0,\ \ \xi^4{}_{yzz} = 0,\ \ \xi^4{}_{zzz} = 0,\ \ \eta_{uu} = 0\\
&3te^{2x}\eta_t + t^2e^{2x}\eta_{tt} - \eta_{zz} - \eta_{yy} - 2e^{2x}\eta_x - e^{2x}\eta_{xx} = 0,\\
&t^2\eta_{tu} + \xi^1{}_x + t^2\xi^4{}_{tz} - \xi^1 + t\xi^1{}_t = 0,\\
&t\eta_{ux} + \xi^1{}_x + t\xi^4{}_{xz} - \xi^1 + t\xi^1{}_t = 0,\\
&\eta_{uy} + \xi^4{}_{yz} - e^{2x}\xi^3{}_x + te^{2x}\xi^3{}_t = 0,\\
&\eta_{uz} + \xi^4{}_{zz} - e^{2x}\xi^4{}_x + te^{2x}\xi^4{}_t = 0,\\
&t\xi^2 + \xi^1 + t\xi^4{}_z - t\xi^1{}_t = 0.
\end{aligned}
$$

Solving this system iteratively gives

$$
\begin{aligned}
\xi^1 &= \tfrac{1}{2}(c_7t^2 - c_8)e^{-x} + \tfrac{1}{2}((c_7y^2 - 2c_5y + c_7z^2 + 2c_{11}z + 2c_{13})t^2\\
&\quad + 2c_{12} - c_8y^2 + 2c_{10}z + 2c_4y - c_8z^2)e^x + c_{14}t,
\end{aligned} \tag{4.66}
$$

$$\xi^2 = \frac{1}{2}\frac{1}{t}((-c_7t^2 - c_8)e^{-x} + ((c_7y^2 - 2c_5y + c_7z^2 + 2c_{11}z + 2c_{13})t^2 \\ + c_8z^2 - 2c_{12} + c_8y^2 - 2c_4y - 2c_{10}z)e^x + 2t(c_1y - c_9 - c_2z)), \tag{4.67}$$

$$\xi^3 = \frac{1}{2}\frac{1}{t}(((2c_7y - 2c_5)t^2 + 2c_8y - 2c_4)e^{-x} + t(c_1e^{-2x} - c_1y^2 \\ + (2c_9 + 2c_2z)y + c_1z^2 + 2c_6 + 2c_3z)), \tag{4.68}$$

$$\xi^4 = \frac{1}{2}\frac{1}{t}(((2c_7z - 2c_{11})t^2 - 2c_{10} + 2c_8z)e^{-x} + t(-c_2e^{-2x} + c_1z^2 \\ - (-2c_9 + 2c_1y)z - c_2y^2 + 2c_{15} - 2c_3y)), \tag{4.69}$$

$$\eta = (-c_7te^{-x} + (-c_7(y^2 + z^2) - 2c_{13} + 2c_5y - c_{11}z)te^x + c_{16})u \\ + \beta(t,x,y,z), \tag{4.70}$$

where β is an arbitrary function satisfying equation (4.63). This yields a 16-dimensional algebra (excluding the infinite symmetry) with basis given by

$$\begin{aligned}
\mathscr{X}_1 &= y\partial_x + \tfrac{1}{2}(e^{-2x} - y^2 + z^2)\partial_y - yz\partial_z, \\
\mathscr{X}_2 &= -z\partial_x + yz\partial_y - \tfrac{1}{2}(e^{-2x} + y^2 - z^2)\partial_z, \\
\mathscr{X}_3 &= z\partial_y - y\partial_z, \\
\mathscr{X}_4 &= ye^x\partial_t - \tfrac{1}{t}ye^x\partial_x - \tfrac{1}{t}e^{-x}\partial_y, \\
\mathscr{X}_5 &= -t^2ye^x\partial_t - tye^x\partial_x - te^{-x}\partial_y + 2tyue^x\partial_u, \\
\mathscr{X}_6 &= \partial_y, \\
\mathscr{X}_7 &= \tfrac{1}{2}t^2(1 + e^{2x}(y^2 + z^2))e^{-x}\partial_t + \tfrac{1}{2}t(-1 + e^{2x}(y^2 + z^2))e^{-x}\partial_x + tye^{-x}\partial_y \\
&\quad + tze^{-x}\partial_z - tu(1 + e^{2x}(y^2 + z^2))e^{-x}\partial_u, \\
\mathscr{X}_8 &= -\tfrac{1}{2}(1 + e^{2x}(y^2 + z^2))e^{-x}\partial_t + \tfrac{1}{2t}(-1 + e^{2x}(y^2 + z^2))e^{-x}\partial_x + \tfrac{1}{t}ye^{-x}\partial_y \\
&\quad + \tfrac{1}{t}ze^{-x}\partial_z, \\
\mathscr{X}_9 &= -\partial_x + y\partial_y + z\partial_z,
\end{aligned}$$

$$\mathscr{X}_{10} = ze^x\partial_t - \tfrac{1}{t}ze^x\partial_x - \tfrac{1}{t}e^{-x}\partial_z,$$

$$\mathscr{X}_{11} = t^2ze^x\partial_t + tze^x\partial_x + te^{-x}\partial_z - 2tzue^x\partial_u,$$

$$\mathscr{X}_{12} = e^x\partial_t - \tfrac{1}{t}e^x\partial_x,$$

$$\mathscr{X}_{13} = t^2e^x\partial_t + te^x\partial_x - 2tue^x\partial_u,$$

$$\mathscr{X}_{14} = t\partial_t,$$

$$\mathscr{X}_{15} = \partial_z,$$

$$\mathscr{X}_{16} = u\partial_u.$$

The commutation relations of the Lie algebra of the 16 Lie symmetries are shown in the following table:

[,]	$\mathscr{X}_1$	$\mathscr{X}_2$	$\mathscr{X}_3$	$\mathscr{X}_4$	$\mathscr{X}_5$	$\mathscr{X}_6$	$\mathscr{X}_7$	$\mathscr{X}_8$	$\mathscr{X}_9$	$\mathscr{X}_{10}$	$\mathscr{X}_{11}$	$\mathscr{X}_{12}$	$\mathscr{X}_{13}$	$\mathscr{X}_{14}$	$\mathscr{X}_{15}$
$\mathscr{X}_1$	0	0	$\mathscr{X}_2$	$-\mathscr{X}_8$	$-\mathscr{X}_7$	$\mathscr{X}_9$	0	0	$-\mathscr{X}_1$	0	0	$\mathscr{X}_4$	$-\mathscr{X}_5$	0	$-\mathscr{X}_3$
$\mathscr{X}_2$	0	0	$-\mathscr{X}_1$	0	0	$-\mathscr{X}_3$	0	0	$-\mathscr{X}_2$	$\mathscr{X}_8$	$-\mathscr{X}_7$	$-\mathscr{X}_{10}$	$-\mathscr{X}_{11}$	0	$-\mathscr{X}_9$
$\mathscr{X}_3$	$-\mathscr{X}_2$	$\mathscr{X}_1$	0	$\mathscr{X}_{10}$	$-\mathscr{X}_{11}$	$\mathscr{X}_{15}$	0	0	0	$-\mathscr{X}_4$	$\mathscr{X}_5$	0	0	0	$-\mathscr{X}_6$
$\mathscr{X}_4$	$\mathscr{X}_8$	0	$-\mathscr{X}_{10}$	0	$2(\mathscr{X}_{14}-\mathscr{X}_{16})$	$-\mathscr{X}_{12}$	$-2\mathscr{X}_1$	0	0	0	$-2\mathscr{X}_3$	0	$-2\mathscr{X}_6$	$\mathscr{X}_4$	0
$\mathscr{X}_5$	$\mathscr{X}_7$	0	$\mathscr{X}_{11}$	$2(\mathscr{X}_{16}-\mathscr{X}_{14})$	0	$\mathscr{X}_{13}$	0	$-2\mathscr{X}_1$	0	$2\mathscr{X}_3$	0	$2\mathscr{X}_6$	0	$-\mathscr{X}_5$	0
$\mathscr{X}_6$	$-\mathscr{X}_9$	$\mathscr{X}_3$	$-\mathscr{X}_{15}$	$\mathscr{X}_{12}$	$-\mathscr{X}_{13}$	0	$-\mathscr{X}_5$	$-\mathscr{X}_4$	$\mathscr{X}_6$	0	0	0	0	0	0
$\mathscr{X}_7$	0	0	0	$2\mathscr{X}_1$	0	$\mathscr{X}_5$	0	0	$-\mathscr{X}_7$	$-2\mathscr{X}_2$	0	$2(\mathscr{X}_{16}-\mathscr{X}_9-\mathscr{X}_{14})$	0	$-\mathscr{X}_7$	$-\mathscr{X}_{11}$
$\mathscr{X}_8$	0	0	0	0	$2\mathscr{X}_1$	$\mathscr{X}_4$	0	0	$-\mathscr{X}_8$	0	$2\mathscr{X}_2$	0	$2(\mathscr{X}_9-\mathscr{X}_{14}+\mathscr{X}_{16})$	$\mathscr{X}_8$	$\mathscr{X}_{10}$
$\mathscr{X}_9$	$\mathscr{X}_1$	$\mathscr{X}_2$	0	0	0	$-\mathscr{X}_6$	$\mathscr{X}_7$	$\mathscr{X}_8$	0	0	0	$-\mathscr{X}_{12}$	$-\mathscr{X}_{13}$	0	$-\mathscr{X}_{15}$
$\mathscr{X}_{10}$	0	$-\mathscr{X}_8$	$\mathscr{X}_4$	0	$-2\mathscr{X}_3$	0	$2\mathscr{X}_2$	0	0	0	$2(\mathscr{X}_{16}-\mathscr{X}_{14})$	0	$-2\mathscr{X}_{15}$	$\mathscr{X}_{10}$	$-\mathscr{X}_{12}$
$\mathscr{X}_{11}$	0	$\mathscr{X}_7$	$-\mathscr{X}_5$	$2\mathscr{X}_3$	0	0	0	$-2\mathscr{X}_2$	0	$2(\mathscr{X}_{14}-\mathscr{X}_{16})$	0	$-2\mathscr{X}_{15}$	0	$-\mathscr{X}_{11}$	$-\mathscr{X}_{13}$
$\mathscr{X}_{12}$	$-\mathscr{X}_4$	$\mathscr{X}_{10}$	0	0	$-2\mathscr{X}_6$	0	$2(\mathscr{X}_9+\mathscr{X}_{14}-\mathscr{X}_{16})$	0	$\mathscr{X}_{12}$	0	$2\mathscr{X}_{15}$	0	0	$\mathscr{X}_{12}$	0
$\mathscr{X}_{13}$	$\mathscr{X}_5$	$\mathscr{X}_{11}$	0	$2\mathscr{X}_6$	0	0	0	$2(\mathscr{X}_{14}-\mathscr{X}_9-\mathscr{X}_{16})$	$\mathscr{X}_{13}$	$2\mathscr{X}_{15}$	0	0	0	$-\mathscr{X}_{13}$	0
$\mathscr{X}_{14}$	0	0	0	$-\mathscr{X}_4$	$\mathscr{X}_5$	0	$\mathscr{X}_7$	$-\mathscr{X}_8$	0	$-\mathscr{X}_{10}$	$\mathscr{X}_{11}$	$-\mathscr{X}_{12}$	$\mathscr{X}_{13}$	0	0
$\mathscr{X}_{15}$	$\mathscr{X}_3$	$\mathscr{X}_9$	$\mathscr{X}_6$	0	0	0	$\mathscr{X}_{11}$	$-\mathscr{X}_{10}$	$\mathscr{X}_{15}$	$\mathscr{X}_{12}$	$\mathscr{X}_{13}$	0	0	0	0
$[\mathscr{X}_{16},\ \mathscr{X}_j] = 0\ \forall\ j = 1,2,\ldots,16$															

Table 4.2: Algebra of Lie symmetries associated with wave equation on Milne metric

The wave equation on Milne metric admits 16 Lie point symmetries which are maximal symmetries in flat space in (3+1) dimensions. Note that in the literature [15] it was shown that the wave equations in plane symmetric static spacetime admits 15 Lie point symmetries which are one less than 16 Lie point symmetries of the wave equation in 3 Cartesian space dimensions suggested in the book of Ibragimov [18] in which he showed that the three-dimensional linear wave equation in Euclidean space admits 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry. As for the Killing symmetries of the Milne metric concerned they are 10. This shows that the Lie symmetries of the wave equation inherit 6 additional symmetries not given by Killing vectors. The fundamental difference between the Lie and the Killing symmetries is that Lie symmetries leave the wave equation on the Milne metric invariant while Killing symmetries are vectors along which Lie derivative of the metric is zero.

Chapter 5

Conclusions and Recommendations

Noether symmetries of the Lagrangian constructed from the Milne metric are found. It is shown that the Noether symmetries of this model give 7 additional conservation laws as compared with the conventional conserved vectors such as Killing vectors. We also found Lie point symmetries of the wave equation on the Milne metric and showed that it admits maximal group of symmetries the same as the three-dimensional linear wave equation in cartesian space.

It suggests that if the Lorentzian metric is flat, both Noether and Lie point symmetry groups will be maximal. It also suggests that Noether symmetries would always provide a larger Lie algebra of which the KVs will form a subalgebra. It is worth mentioning here that an n-dimensional flat space admits a total number of $\frac{n(n-1)}{2}+2n+3=\frac{n^2+3n+6}{2}$ symmetries.

It will be interesting to extend such investigations to more general Lorentzian metrics when they are genuinely non-flat. It is hoped that such investigations will add to one's understanding of Lie and Noether point symmetries of the wave equation and other equations of interest in such geometries.

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Vitae

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